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A Fast-Decodable, Quasi-Orthogonal Space–Time Block Code for 4×2 MIMO

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***Abstract*— In this paper we present a new family of full-rate space–time block codes for 4×2 MIMO. We show how, by combining algebraic and quasi-orthogonal properties of the code, reduced-complexity maximum-likelihood decoding is made possible. In particular, the sphere decoder search can be reduced from a 16- to a 12-dimensional space. Within this family, we found a code that outperforms all previously proposed codes for 4×2 MIMO.**[1]

I. Introduction

In a recent paper, Paredes *et al.* [1] have shown how to construct a family of fast-decodable, full-rate, full-rank space–time block codes (STBCs) for 2×2 MIMO. The maximum-likelihood (ML) decoder can be simplified to a 4-dimensional sphere decoder followed by an Alamouti detector [2]. The best code within this family coincides with a code originally found by Hottinen and Tirkkonen [3], and recently independently rediscovered in [4].

Motivated by the above, we extend the concept of fast-decodable STBC code design to 4×2 MIMO. In particular, we present a new family of full-rate STBC that combines algebraic and quasi-orthogonal structures and enables a complexity reduction of its ML decoder. Note that in the 4×2 case, the full-rank assumption is dropped. At the receiver, only a 12-dimensional real sphere decoder (SD) is needed to conduct the ML search, rather than the standard 16-dimensional SD. Finally, within this family, we found a code that outperforms all previously proposed 4×2 STBCs for 4-QAM signal constellation.

The balance of this paper is organized as follows. Section II introduces system model and code design criteria. In Section III, we give a brief review of the known fast-decodable STBCs for 2×2 MIMO. In Section IV, we present the new fast-decodable STBC for 4×2 MIMO. In Section V, the corresponding fast decoding algorithm is exhibited. Finally, conclusions are drawn in Section VI.

A. Notations

The following notations will be adopted: Boldface letters are used for column vectors, and capital boldface letters for matrices. Superscripts ^{T}, †, and * denote transposition, Hermitian transposition, and complex conjugation, respectively. $\mathbb{Z}$, $\mathbb{C}$, and $\mathbb{Z}[j]$ denote the ring of rational integers, the field of complex numbers, and the ring of Gaussian integers, respectively, where $j^2 = -1$. Also, $\mathbf{I}_n$ denotes the $n \times n$ identity matrix, and $\mathbf{0}_{m \times n}$ denotes the $m \times n$ matrix all of whose elements are 0.

Given a complex number x, we define the $\tilde{(\cdot)}$ operator from $\mathbb{C}$ to $\mathbb{R}^2$ as

$$\tilde{x} \triangleq [\Re(x), \Im(x)]$$

where $\Re(\cdot)$ and $\Im(\cdot)$ denote the real and imaginary parts of a complex number. The $\tilde{(\cdot)}$ operator can be extended to complex vectors $\mathbf{x} = [x_1, \dots x_n] \in \mathbb{C}^n$ as

$$\tilde{\mathbf{x}} \triangleq [\tilde{x}_1, \dots \tilde{x}_n]^T$$

where $(\cdot)^T$ denotes vector transposition. The $\check{(\cdot)}$ operator from $\mathbb{C}$ to $\mathbb{R}^{2\times 2}$ is defined as

$$\check{x} \triangleq \begin{bmatrix} \Re(x) & -\Im(x) \\ \Im(x) & \Re(x) \end{bmatrix}$$

The $\check{(\cdot)}$ operator can be similarly extended to matrices so that a complex matrix times a complex vector can be equivalently written as

$$\widetilde{\mathbf{A}\mathbf{x}} = \check{\mathbf{A}}\tilde{\mathbf{x}}$$

The $\text{vec}(\cdot)$ operator stacks the m column vectors of a $n \times m$ complex matrix into a mn complex column vector. The $\|\cdot\|$ operation denotes the Euclidean norm of a vector, and the Frobenius norm of a matrix. Finally, the Hermitian inner product of two complex column vectors $\mathbf{a}$ and $\mathbf{b}$ is denoted by

$$\langle \mathbf{a}, \mathbf{b} \rangle \triangleq \mathbf{a}^T \mathbf{b}^*$$

II. System Model and Code Design Criteria

We consider a $n_t \times n_r$ MIMO system over block fading channels. At discrete time t, the received signal matrix $\mathbf{Y} \in \mathbb{C}^{n_r \times T}$ is given by

$$\mathbf{Y} = \mathbf{H}\mathbf{X} + \mathbf{N}, \tag{1}$$

where $\mathbf{X} \in \mathbb{C}^{n_t \times T}$ is the codeword matrix, transmitted over T channel uses. Moreover, $\mathbf{N} \in \mathbb{C}^{n_r \times T}$ is a complex white Gaussian noise with i.i.d. entries $\sim \mathcal{N}_{\mathbb{C}}(0, N_0)$, and $\mathbf{H} =$

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$[h_{i\ell}] \in \mathbb{C}^{n_r \times n_t}$ is the channel matrix, assumed to remain constant during the transmission of a codeword, and to take on independent values from codeword to codeword. The elements of $\mathbf{H}$ are assumed to be i.i.d. circularly symmetric Gaussian random variables $\sim \mathcal{N}_{\mathbb{C}}(0,1)$. The realization of $\mathbf{H}$ is assumed to be known at the receiver, but not at the transmitter.

The following definitions are relevant here:

Definition 1: ***(Code rate)*** The code rate of a STBC is defined as the number κ of independent information symbols per codeword, drawn from a complex constellation $\mathbb{S}$. If $\kappa = n_r T$, the STBC is said to have *full rate*. □

Consider now ML decoding. This consists of finding the code matrix $\mathbf{X}$ that achieves the minimum of the squared norm $m(\mathbf{X}) \triangleq \|\mathbf{Y} - \mathbf{H}\mathbf{X}\|^2$.

Definition 2: ***(Decoding Complexity)*** The ML decoding complexity can be measured by counting the minimum number of values of $m(\mathbf{X})$ that should be computed in ML decoding. This number cannot exceed M^κ, with $M = |\mathbb{S}|$, the worst-case decoding complexity achieved by an exhaustive-search ML decoder. □

Definition 3: ***(Simplified decoding)*** We say that a STBC admits simplified decoding if ML decoding can be achieved with less than M^κ computations of $m(\mathbf{X})$. □

Assuming that the codeword $\mathbf{X}$ is transmitted, it may occur that $\|\mathbf{Y} - \mathbf{H}\mathbf{X}\|^2 > \|\mathbf{Y} - \mathbf{H}\widehat{\mathbf{X}}\|^2$, with $\widehat{\mathbf{X}} \neq \mathbf{X}$, resulting in a *pairwise error*. Let r denote the rank of the *codeword-difference matrix* $\mathbf{X} - \widehat{\mathbf{X}}$, with $\widehat{\mathbf{X}} \neq \mathbf{X}$, and let $\mathbf{E} \triangleq (\mathbf{X} - \widehat{\mathbf{X}})(\mathbf{X} - \widehat{\mathbf{X}})^\dagger$ be the *codeword-distance matrix*. Let δ denote the product of non-zero eigenvalues of the codeword distance matrix $\mathbf{E}$. The error probability of a STBC is upper-bounded by the following union bound:

$$\begin{aligned} P(e) &\leq \frac{1}{M^\kappa} \sum_{\mathbf{X}} \sum_{\mathbf{X} \neq \hat{\mathbf{X}}} P(\mathbf{X} \to \hat{\mathbf{X}}) \\ &= \frac{1}{M^\kappa} \sum_{r} \sum_{\delta} A(r,\delta) P(r,\delta) \end{aligned} \tag{2}$$

where $P(\mathbf{X} \to \hat{\mathbf{X}})$ denotes the pairwise error probability (PEP) among all distinct $(\mathbf{X}, \hat{\mathbf{X}})$. The term $P(r,\delta)$ represents the PEP of the codewords with rank r and eigenvalue product δ, while $A(r,\delta)$ denotes the associated multiplicity.

Definition 4: ***(Full-diversity STBC)*** A full-diversity STBC is one with $r = n_t$ over all possible codeword-difference matrices. □

For a full-diversity STBC, the worst-case PEP depends asymptotically, for high signal-to-noise ratios, on both the rank $r = n_t$ and the *minimum determinant* of the codeword distance matrix

$$\delta_{\min} \triangleq \min_{\mathbf{X} \neq \hat{\mathbf{X}}} \det(\mathbf{E})$$

The "rank-and-determinant criterion" (RDC) of code design requires the maximization of both r and $\delta_{\min}$. This criterion yields *diversity gain* $n_r n_t$ and *coding gain* $(\delta_{\min})^{1/n_t}$ [5].

For a non full-diversity STBC, the minimum determinant equals to zero. In such a case, we have to minimize the associated multiplicity of the *dominant pairwise terms* of rank $r \leq n_t$ independently of their product distance.

III. Fast-decodable codes for 2×2 MIMO

Consider now 2×2 STBCs. These are full-rate and full-diversity if $\kappa = 4$ symbols/codeword, and $r = n_t$.

Definition 5: ***(Fast-decodable STBCs for*** 2×2 ***MIMO)*** A 2×2 STBC allows fast ML decoding if its complexity does not exceed $2M^3$. □

Here we examine 2×2 fast-decodable STBCs endowed with the following structure [3]:

$$\mathbf{X} = \mathbf{X}_a(x_1, x_2) + \mathbf{T}\mathbf{X}_b(z_1, z_2) \tag{3}$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad \mathbf{X}_a(x_1, x_2) = \begin{bmatrix} x_1 & -x_2^* \\ x_2 & x_1^* \end{bmatrix} \tag{4}$$

is an Alamouti 2×2 space–time block codeword [2], and $x_1, x_2 \in \mathbb{Z}[j]$. Moreover, we have

$$\mathbf{X}_b(z_1, z_2) = \begin{bmatrix} z_1 & -z_2^* \\ z_2 & z_1^* \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} = \mathbf{U} \begin{bmatrix} x_3 \\ x_4 \end{bmatrix} \tag{5}$$

where $z_1, z_2 \in \mathbb{C}$, $x_3, x_4 \in \mathbb{Z}[j]$, and $\mathbf{U} \in \mathbb{C}^{2 \times 2}$ is the unitary matrix

$$\mathbf{U} = \begin{bmatrix} \varphi_1 & -\varphi_2^* \\ \varphi_2 & \varphi_1^* \end{bmatrix}$$

with $|\varphi_1|^2 + |\varphi_2|^2 = 1$. Vectorizing, and separating real and imaginary parts, the matrices $\mathbf{X}$ yield

$$\widetilde{\text{vec}(\mathbf{X})} = \mathbb{G}_1[\tilde{x}_1, \tilde{x}_2]^T + \mathbb{G}_2[\tilde{x}_3, \tilde{x}_4]^T$$

where $\mathbb{G}_1, \mathbb{G}_2 \in \mathbb{R}^{8 \times 4}$ are the *generator matrices* of $\mathbf{X}_a$ and $\mathbf{T}\mathbf{X}_b$, respectively. Note that the matrix $\mathbf{T}$ is chosen in order to guarantee that the subspace spanned by the columns of $\mathbf{R}_2$ is orthogonal to the one spanned by the columns of $\mathbf{R}_1$. This implies that the code has *cubic shaping* (or that is *information lossless*).

The matrix $\mathbf{U}$ is chosen in order to achieve full rank and maximize the minimum determinant. The best code of the form (3) was first proposed in [3] under the name *twisted space–time transmit diversity* code, and recently rediscovered independently in [1]. It is characterized by the following choice of the unitary matrix $\mathbf{U}$:

$$\mathbf{U} = \frac{1}{\sqrt{7}} \begin{bmatrix} 1+j & -1+2j \\ 1+2j & 1-j \end{bmatrix}$$

This code was also found in [4] by numerical optimization, and classified under the rubric of *multi-strata* space–time codes. This code has minimum determinant $\delta_{\min} = 16/7$ for 4-QAM signalling, which is smaller than the Golden code ($\delta_{\min} = 16/5$) [6].

At the receiver, due to the linearity of the code, a sphere decoder can be employed. It was pointed out both in [1] and in [4] that the code in (3) admits a low-complexity decoder thanks to orthogonality properties of the two component codes in (3).

IV. New STBC for 4×2 MIMO Systems

Here we design a fast-decodable 4×2 STBC based on the concepts elaborated upon in the previous sections. We first introduce the relevant definitions.

Definition 6: ***(Quasi-orthogonal structure)*** [11] A code such that

$$\mathbf{X} = \begin{bmatrix} x_1 & -x_2^* & -x_3^* & x_4 \\ x_2 & x_1^* & -x_4^* & -x_3 \\ x_3 & -x_4^* & x_1^* & -x_2 \\ x_4 & x_3^* & x_2^* & x_1 \end{bmatrix}$$

where $x_i \in \mathbb{C}$, $i = 1, \ldots, 4$, is said to have a quasi-orthogonal structure. Note that quasi-orthogonal STBCs are not full rank, and $r = 2$. □

Definition 7: ***(Full-rate, fast-decodable STBC for*** 4×2 ***MIMO)*** A full-rate, fast-decodable STBC for 4×2 MIMO, denoted $\mathcal{G}'$, has $\kappa = 8$ symbols/codeword, and can be decoded by a 12-dimensional real SD algorithm (rather than the standard 16-dimensional SD). □

The 4×4 codeword matrix $\mathbf{X} \in \mathcal{G}'$ encodes eight QAM symbols $\mathbf{x} = [x_1, \ldots, x_8] \in \mathbb{Z}[j]$, and is transmitted by using the channel four times, i.e., $T = 4$. Following the idea of the previous section, we choose the following codeword structure:

$$\mathbf{X} = \mathbf{X}_a(x_1, x_2, x_3, x_4) + \mathbf{T}\mathbf{X}_b(z_1, z_2, z_3, z_4) \qquad (6)$$

where

$$\mathbf{T} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \qquad (7)$$

is used to preserve the orthogonality between the two components of the code (similarly to the codes of previous section), and

$$\mathbf{X}_a(x_1, x_2, x_3, x_4) = \begin{bmatrix} x_1 & -x_2^* & -x_3^* & x_4 \\ x_2 & x_1^* & -x_4^* & -x_3 \\ x_3 & -x_4^* & x_1^* & -x_2 \\ x_4 & x_3^* & x_2^* & x_1 \end{bmatrix} \qquad (8)$$

follows the quasi-orthogonal STBC structure of [11], where $x_1, x_2, x_3, x_4 \in \mathbb{Z}[j]$. The remaining matrix in (6) is defined as

$$\mathbf{X}_b(z_1, z_2, z_3, z_4) = \begin{bmatrix} z_1 & -z_2^* & -z_3^* & z_4 \\ z_2 & z_1^* & -z_4^* & -z_3 \\ z_3 & -z_4^* & z_1^* & -z_2 \\ z_4 & z_3^* & z_2^* & z_1 \end{bmatrix} \qquad (9)$$

with

$$\begin{bmatrix} z_1 \\ z_2 \\ z_3 \\ z_4 \end{bmatrix} = \mathbf{U} \begin{bmatrix} x_5 \\ x_6 \\ x_7 \\ x_8 \end{bmatrix} \qquad (10)$$

where $z_i \in \mathbb{C}$, $x_k \in \mathbb{Z}[j]$, $i = 1, \ldots, 4$, $k = 5, \ldots, 8$ and

$$\mathbf{U} = [\boldsymbol{\varphi}_1 | \boldsymbol{\varphi}_2 | \boldsymbol{\varphi}_3 | \boldsymbol{\varphi}_4] = \begin{bmatrix} \varphi_{11} & \varphi_{12} & \varphi_{13} & \varphi_{14} \\ \varphi_{21} & \varphi_{22} & \varphi_{23} & \varphi_{24} \\ \varphi_{31} & \varphi_{32} & \varphi_{33} & \varphi_{34} \\ \varphi_{41} & \varphi_{42} & \varphi_{43} & \varphi_{44} \end{bmatrix} \qquad (11)$$

is a 4×4 unitary matrix. Note that 1) The matrix $\mathbf{T}$ guarantees cubic shaping, and 2) Since the matrix $\mathbf{X}_a$ has a quasi-orthogonal structure, the code is not full rank: in fact, it has $r = 2$. As a consequence, we conduct a search over the matrices $\mathbf{U}$ leading to the minimum of $\sum_\delta A(2, \delta)$. The term $A(2, \delta)$ represents the total number of codeword difference matrices of rank 2 and product distance δ. Since an exhaustive search through all 4×4 unitary matrices is too complex, we focus on those with the form

$$\mathbf{U} = \mathbf{D}\mathbf{F} \qquad (12)$$

where $\mathbf{F} \triangleq [\exp(j2\pi\ell n/4)]_{\ell,n=1,\ldots,4}$ is a 4×4 discrete-Fourier-transform matrix, and $\mathbf{D} \triangleq \mathrm{diag}(\exp(j2\pi n_\ell/N))$ for some integers N, ℓ, with $0 \leq n_\ell < N$ and $\ell = 1, \ldots, 4$.

For 4-QAM signaling, taking $N = 7$ and $n_\ell = 1, 2, 5, 6$, we have obtained

$$\mathbf{U} = \begin{bmatrix} 0.31 + 0.39i & 0.31 + 0.39i & 0.31 + 0.39i & 0.31 + 0.39i \\ -0.11 + 0.49i & -0.49 - 0.11i & 0.11 - 0.49i & 0.49 + 0.11i \\ -0.11 - 0.49i & 0.11 + 0.49i & -0.11 - 0.49i & 0.11 + 0.49i \\ 0.31 - 0.39i & -0.39 - 0.31i & -0.31 + 0.39i & 0.39 + 0.31i \end{bmatrix}$$

which yields the minimum $\sum_\delta A(2, \delta)$.

Under 4-QAM signaling, we compare the minimum determinants δ_{min} and their associated multiplicities $A(r, \delta_{min})$, as well as the codeword-error rates (CERs) of the above STBC to the following 4×2 codes:

1) Code with the structure (6), with $\mathbf{U}$ the 4×4 "perfect" rotation matrix [12].
2) The best DjABBA code of [3].
3) The "perfect" two-layer code of [13].

Determinant and multiplicity values are shown in Table I. It can be seen that the proposed 4×2 STBC has the minimum $\sum_\delta A(2, \delta)$, when compared to the rank-2 code with perfect rotation matrix $\mathbf{U}$ in [12]. The CERs are shown in Fig. 1. The proposed code achieves the best CER up to 10^{-5}. Due to diversity loss, the performance curve of the new code and the one of DjABBA cross over at CER$= 10^{-5}$.

Codes	$\delta_{\min}$	Multiplicity
New STBC	0	$\sum_\delta A(2, \delta) = 160$
Perfect Code $\mathbf{U}$ matrix	0	$\sum_\delta A(2, \delta) = 560$
DjABBA	0.8304	$A(4, 0.8304) = 770$
Two-Layers Perfect Code	0.0016	$A(4, 0.0016) = 128$

TABLE I
Minimum determinants of 4×2 STBCs with 4-QAM signaling

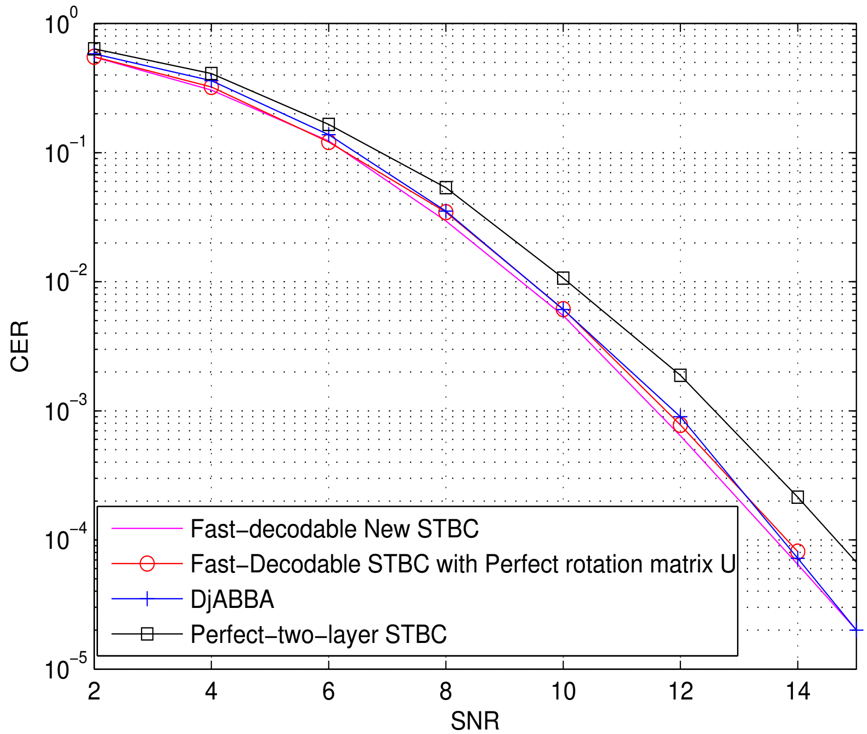

Fig. 1. Comparison of the CER of different 4×2 STBCs with 4-QAM signaling.

V. Low-Complexity Decoding for 4×2 MIMO

In this section, we first analyze the sphere decoding process, next we discuss the complexity reduction.

A. *Sphere Decoding for 4×2 MIMO*

Let $\mathbf{Y} = [y_{\ell n}] \in \mathbb{C}^{2\times 4}$, $\mathbf{H} = [h_{\ell n}] \in \mathbb{C}^{2\times 4}$, and $\mathbf{N} = [n_{\ell n}] \in \mathbb{C}^{2\times 4}$. After vectorization, we obtain

$$\mathbf{y} = \mathcal{H}\mathbf{x} + \mathbf{n} \tag{13}$$

where

$$\begin{aligned} \mathbf{y} &\triangleq [y_{11}, y_{12}^*, y_{13}^*, y_{14}, y_{21}, y_{22}^*, y_{23}^*, y_{24}]^T \\ \mathbf{n} &\triangleq [n_{11}, n_{12}^*, n_{13}^*, n_{14}, n_{21}, n_{22}^*, n_{23}^*, n_{24}]^T \\ \mathbf{x} &\triangleq [x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8]^T \end{aligned}$$

and

$$\mathcal{H} \triangleq [\mathbf{F}_1 | \mathbf{F}_2] \tag{14}$$

where

$$\mathbf{F}_1 = [\mathbf{f}_1|\mathbf{f}_2|\mathbf{f}_3|\mathbf{f}_4] = \begin{bmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{12}^* & -h_{11}^* & h_{14}^* & -h_{13}^* \\ h_{13}^* & h_{14}^* & -h_{11}^* & -h_{12}^* \\ h_{14} & -h_{13} & -h_{12} & h_{11} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{22}^* & -h_{21}^* & h_{24}^* & -h_{23}^* \\ h_{23}^* & h_{24}^* & -h_{21}^* & -h_{22}^* \\ h_{24} & -h_{23} & -h_{22} & h_{21} \end{bmatrix}$$

and $\mathbf{F}_2 = [\mathbf{f}_5|\mathbf{f}_6|\mathbf{f}_7|\mathbf{f}_8]$ with

$$\mathbf{f}_5 = \mathbf{M}\boldsymbol{\varphi}_1 \quad \mathbf{f}_6 = \mathbf{M}\boldsymbol{\varphi}_2 \quad \mathbf{f}_7 = \mathbf{M}\boldsymbol{\varphi}_3 \quad \mathbf{f}_8 = \mathbf{M}\boldsymbol{\varphi}_4$$

where

$$\mathbf{M} = \begin{bmatrix} h_{11} & h_{12} & -h_{13} & -h_{14} \\ h_{12}^* & -h_{11}^* & -h_{14}^* & h_{13}^* \\ -h_{13}^* & -h_{14}^* & -h_{11}^* & -h_{12}^* \\ -h_{14} & h_{13} & -h_{12} & h_{11} \\ h_{21} & h_{22} & -h_{23} & -h_{24} \\ h_{22}^* & -h_{21}^* & -h_{24}^* & h_{23}^* \\ -h_{23}^* & -h_{24}^* & -h_{21}^* & -h_{22}^* \\ -h_{24} & h_{23} & -h_{22} & h_{21} \end{bmatrix}$$

We conduct the QR decomposition of $\mathcal{H}$, i.e., $\mathcal{H} = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{C}^{8\times 8}$ is an unitary matrix and $\mathbf{R} \in \mathbb{C}^{8\times 8}$ is an upper-triangular matrix. Here $\mathbf{Q}$ and $\mathbf{R}$ are given by

$$\begin{aligned} \mathbf{Q} &= [\mathbf{e}_1|\mathbf{e}_2|\mathbf{e}_3|\mathbf{e}_4|\mathbf{e}_5|\mathbf{e}_6|\mathbf{e}_7|\mathbf{e}_8] \\ \mathbf{R} &= \begin{bmatrix} \|\mathbf{d}_1\| & \langle \mathbf{f}_2, \mathbf{e}_1 \rangle & \langle \mathbf{f}_3, \mathbf{e}_1 \rangle & \cdots & \langle \mathbf{f}_8, \mathbf{e}_1 \rangle \\ 0 & \|\mathbf{d}_2\| & \langle \mathbf{f}_3, \mathbf{e}_2 \rangle & \cdots & \langle \mathbf{f}_8, \mathbf{e}_2 \rangle \\ 0 & 0 & \|\mathbf{d}_3\| & \cdots & \langle \mathbf{f}_8, \mathbf{e}_3 \rangle \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \|\mathbf{d}_8\| \end{bmatrix} \end{aligned}$$

The QR decomposition is related to the Gram–Schmidt orthogonalization algorithm through the following equations:

$$\mathbf{u}_1 = \mathbf{f}_1, \quad \mathbf{e}_1 = \frac{\mathbf{u}_1}{\|\mathbf{u}_1\|}$$

$$\mathbf{u}_i = \mathbf{f}_i - \sum_{j=1}^{i-1} \mathrm{Proj}_{\mathbf{e}_j}\mathbf{f}_i, \quad \mathbf{e}_i = \frac{\mathbf{u}_i}{\|\mathbf{u}_i\|}, \quad i = 2, \cdots, 8$$

where $\mathrm{Proj}_{\mathbf{u}}\mathbf{v} \triangleq \frac{\langle \mathbf{v},\mathbf{u}\rangle}{\langle \mathbf{u},\mathbf{u}\rangle}\mathbf{u}$. Direct computation shows that $\mathbf{R}$ has the following properties:

1) $\langle \mathbf{f}_2, \mathbf{e}_1\rangle = 0$
2) $\mu \triangleq \|\mathbf{d}_i\|^2 = \sum_{i=1}^{2}\sum_{j=1}^{4}|h_{ij}|^2$
3) $\langle \mathbf{f}_4, \mathbf{e}_1\rangle = -\langle \mathbf{f}_3, \mathbf{e}_2\rangle = \Phi_1/\sqrt{\mu}$ where $\Phi_1 \triangleq 2\Re(h_{11}^* h_{14} + h_{21}h_{24}^* - h_{12}h_{13}^* - h_{22}h_{23}^*)$
4) $\gamma \triangleq \|\mathbf{d}_i\|^2 = \mu - \Phi_1^2/\mu$ with $i = 3, 4$
5)

$$\mathbf{R} = \begin{bmatrix} \mathbf{S}_1 & \mathbf{S}_2 \\ \mathbf{0}_{4\times 4} & \mathbf{S}_3 \end{bmatrix} \tag{15}$$

where

$$\mathbf{S}_1 = \begin{bmatrix} \sqrt{\mu} & 0 & 0 & \Phi_1 \\ 0 & \sqrt{\mu} & -\Phi_1 & 0 \\ 0 & 0 & \sqrt{\gamma} & 0 \\ 0 & 0 & 0 & \sqrt{\gamma} \end{bmatrix} \tag{16}$$

$$\mathbf{S}_2 = \begin{bmatrix} \langle \mathbf{f}_5, \mathbf{e}_1\rangle & \langle \mathbf{f}_6, \mathbf{e}_1\rangle & \langle \mathbf{f}_7, \mathbf{e}_1\rangle & \langle \mathbf{f}_8, \mathbf{e}_1\rangle \\ \langle \mathbf{f}_5, \mathbf{e}_2\rangle & \langle \mathbf{f}_6, \mathbf{e}_2\rangle & \langle \mathbf{f}_7, \mathbf{e}_2\rangle & \langle \mathbf{f}_8, \mathbf{e}_2\rangle \\ \langle \mathbf{f}_5, \mathbf{e}_3\rangle & \langle \mathbf{f}_6, \mathbf{e}_3\rangle & \langle \mathbf{f}_7, \mathbf{e}_3\rangle & \langle \mathbf{f}_8, \mathbf{e}_3\rangle \\ \langle \mathbf{f}_5, \mathbf{e}_4\rangle & \langle \mathbf{f}_6, \mathbf{e}_4\rangle & \langle \mathbf{f}_7, \mathbf{e}_4\rangle & \langle \mathbf{f}_8, \mathbf{e}_4\rangle \end{bmatrix} \tag{17}$$

and

$$\mathbf{S}_3 = \begin{bmatrix} \|\mathbf{d}_5\| & \langle \mathbf{f}_6, \mathbf{e}_5\rangle & \langle \mathbf{f}_7, \mathbf{e}_5\rangle & \langle \mathbf{f}_8, \mathbf{e}_5\rangle \\ 0 & \|\mathbf{d}_6\| & \langle \mathbf{f}_7, \mathbf{e}_6\rangle & \langle \mathbf{f}_8, \mathbf{e}_6\rangle \\ 0 & 0 & \|\mathbf{d}_7\| & \langle \mathbf{f}_8, \mathbf{e}_7\rangle \\ 0 & 0 & 0 & \|\mathbf{d}_8\| \end{bmatrix} \tag{18}$$

□

The QR decomposition allows us to rewrite (13) as

$$\mathbf{y} = \mathcal{H}\mathbf{x} + \mathbf{n} = \mathbf{QRx} + \mathbf{n} \tag{19}$$

Premultiplication of (19) by $\mathbf{Q}^\dagger$ yields

$$\mathbf{r} = \mathbf{Q}^\dagger\mathbf{y} = \mathbf{Rx} + \mathbf{w} \tag{20}$$

Let $\mathbf{r} = [r_1, \ldots, r_8]^T$, and $\mathbf{w} = \mathbf{Q}^\dagger\mathbf{n} = [w_1, \ldots, w_8]^T$. Separating real and imaginary parts in (20), we obtain

$$\tilde{\mathbf{r}} = \check{\mathbf{R}}\tilde{\mathbf{x}} + \tilde{\mathbf{w}} \Longrightarrow \mathbf{v} = \Theta\mathbf{u} + \tilde{\mathbf{w}} \tag{21}$$

where $\mathbf{v} \triangleq [v_1, \ldots, v_{16}]^T = \tilde{\mathbf{r}}$, $\mathbf{u} \triangleq [u_1, \ldots, u_{16}]^T = \tilde{\mathbf{x}}$, $\tilde{\mathbf{w}} \triangleq [\tilde{w}_1, \ldots, \tilde{w}_8]^T$, and $\Theta \triangleq (\theta_{ij}) \triangleq \check{\mathbf{R}}$, $i, j = 1, \ldots, 16$.

We now apply SD after restricting $\mathcal{S}$ to square QAM constellations, i.e., assuming $u_i \in \mathcal{X}$, where $\mathcal{X}$ is a PAM constellation, so that $\mathcal{S} = \mathcal{X}^2$. The sphere detector finds

$$\hat{\mathbf{u}} = \arg\min_{\mathbf{u}\in\mathcal{X}} \|\mathbf{v} - \Theta\mathbf{u}\|^2 \tag{22}$$

where $\hat{\mathbf{u}} = \{\hat{u}_i\}$, with $i = 1, \ldots, 16$ and $\hat{u}_i \in \mathcal{X}$. It was pointed out in [10] that the search procedure of a SD can be visualized as a bounded tree search. If *standard* real SD is used for 4×2 STBCs, the decoding tree has 16 levels. With our code, the structure of the codeword matrix allows us to use only a 12-level tree search, as shown in the following.

Let us define $\mathbf{u}_i^k \triangleq [u_i, \ldots, u_k]^T$, $i < k$, as the partial symbol vector labeling the path connecting level i to level k. In our case, SD is only used to search the branches corresponding to $\mathbf{u}_5^{16}$, while the symbols in $\mathbf{u}_1^4$ are decoded as in an Alamouti code. We summarize this complexity reduction saying that a 12-dimensional real SD can be used *in lieu* of a 16-dimensional real SD.

Consider again a Schnorr-Euchner (SE) enumeration [9] to decode $\mathbf{u}_5^8$. This starts at level i:

$$S_i(u_i) = \lfloor (v_i - \xi_i)/\theta_{ii} \rceil \in \mathcal{X} \quad i = 16, \ldots, 9 \tag{23}$$

where $\lfloor\cdot\rceil$ denotes the closest integer, $\xi_{16} = 0$, $S_i(u_i)$ is the ZF-DFE component, and ξ_i is the *interference term* on level i from upper level j.

We define the *interference term* on level i from upper levels as

$$\xi_i \triangleq \sum_{j=i+1}^{16} \theta_{ij}u_j, \quad i = 16, \ldots, 9 \tag{24}$$

Since $\mathbf{S}_3$ is not the null matrix, we have nonzero interference terms. The SE algorithm visits the neighbors of the mid-point in a zig-zag order. Let us define

$$\Delta_i \triangleq \mathrm{sign}(v_i - \xi_i - \theta_{ii}u_i)$$

where $\mathrm{sign}(a) = +1$ for $a \geq 0$; otherwise, $\mathrm{sign(a)} = -1$.

SE enumeration is used to search

$$u_i = \{S_i(u_i), S_i(u_i) + \Delta_i, S_i(u_i) - \Delta_i, \ldots\} \subset \mathcal{X}$$

In this tree search, a branch at level $i \in [9, 16]$ contributes to the ML metric by the amount

$$d_i(\mathbf{u}_9^{16}) \triangleq |v_i - \xi_i - \theta_{ii}u_i|^2 \quad 9 \leq i \leq 16 \tag{25}$$

The corresponding path metric is given by

$$T_{i-1} \triangleq \sum_{j=i}^{16} d_j(\mathbf{u}_j^{16}) \tag{26}$$

When the SD reaches the decoding-tree levels $i \leq 8$, a first complexity reduction is available. Given the vector $\mathbf{u}_9^{16}$, ξ_i can be computed for all the remaining levels $i = 1, \ldots, 8$:

$$\xi_i \triangleq \sum_{j=9}^{16} \theta_{ij}u_j$$

which saves a few multiplications in the computation of ξ_i in (24). Next, we proceed with the standard SD searching procedure to find $\mathbf{u}_5^8$. We obtain the remaining symbols as

$$\begin{aligned} S_1(u_1) &= \left\lfloor \frac{(v_1 - \xi_1 - \frac{\Phi_1}{\sqrt{\mu}} u_7)}{\mu} \right\rceil \in \mathcal{X} \\ S_2(u_2) &= \left\lfloor \frac{(v_2 - \xi_2 - \frac{\Phi_1}{\sqrt{\mu}} u_8)}{\mu} \right\rceil \in \mathcal{X} \\ S_3(u_3) &= \left\lfloor \frac{(v_3 - \xi_3 + \frac{\Phi_1}{\sqrt{\mu}} u_5)}{\mu} \right\rceil \in \mathcal{X} \\ S_4(u_4) &= \left\lfloor \frac{(v_4 - \xi_4 + \frac{\Phi_1}{\sqrt{\mu}} u_6)}{\mu} \right\rceil \in \mathcal{X} \end{aligned} \tag{27}$$

We say that the remaining four-level tree search in SD is not necessary, or, equivalently, that a 12-dimensional real SD replaces the standard 16-dimensional one.

At this point, we have a valid vector $\mathbf{u} = [\mathbf{u}_1^8, \mathbf{u}_9^{16}]$. We then compute the corresponding branch and path metrics in (25) and (26), respectively. This completes the search of one path in the 12-dimensional bounded tree. The detailed decoding algorithm is given below.

1) (Input) Input Φ_1 and α.
2) (Initialization) Set $i = 16$, $T_{16} = 0$, $\xi_{16} = 0$, and $d_c = C_0$ (current squared radius of the sphere).
3) Set $u_i = \lfloor (v_i - \xi_i)/\theta_{ii} \rceil$ and $\Delta_i = \mathrm{sign}(v_i - \xi_i - \theta_{ii} u_i)$.
4) (Main step of SD) If $d_c < T_i + |v_i - \xi_i - \theta_{ii} u_i|^2$, then go to Step 5 (outside of the sphere).
   Else if u_i is not in $\mathcal{X}$ go to Step 7 (inside of the sphere, outside of the signal set).
   Else (inside of the sphere, inside of the signal set)
   - If $i \geq 9$ then $\{$ $T_{i-1} = T_i + |r_i u - \xi_i - \theta_{ii} u_i|^2, \xi_{i-1} = \sum_{j=i+1}^{16} \theta_{ij} u_j$, $i = i - 1$ go to Step 3 $\}$.
   - Else if $(i \geq 5)$ then $\{$ $T_{i-1} = T_i + |v_i u - \xi_i - \theta_{ii} u_i|^2, \xi_{i-1} = \sum_{j=9}^{16} \theta_{ij} u_j$, $i = i - 1$ go to Step 3$\}$.
   - Else $(i = 4)$ $\{$Compute u_k using (27) and $T_{k-1} = T_k + |v_k - \sum_{j=k}^{m} \theta_{kj} u_j|^2, k = i, \ldots, 1$, then go to Step 6$\}$.
5) If $i = 16$ then terminate; else set $i = i + 1$ and go to Step 7.
6) (A valid vector is found) Let $d_c = T_0$, save $\hat{\mathbf{u}} = \mathbf{u}$. Then $i = i + 1$ go to Step 7.
7) (SE enumeration of level i) Let $u_i = u_i + \Delta_i$, $\Delta_i = -\Delta_i - \mathrm{sign}(\Delta_\mathrm{i})$, go to Step 4.

B. Complexity Reduction

Summarizing, we observe the following reductions of decoding complexity:

- We use a 12-dimensional real SD to find $\mathbf{u}_5^{16}$. Then, we subtract the interference terms from $\mathbf{u}_5^{16}$ (see (27)). Finally, the partial symbol vector $\mathbf{u}_1^4$ can be computed directly. We see that the standard 16-dimensional real SD is not necessary.
- The interference term ξ_i at level i, $i = 1, \ldots, 8$, admits simple calculation.

In other terms, we observe that the worst-case decoding complexity of fast-decodable STBCs is $2M^7$, as compared to a standard SD complexity M^8. This is due to the fact that:

- A 12-dimensional real SD (6-dimensional complex SD) requires M^6 branch metric computations,
- In each branch of the 6-dimensional tree, part of decoding can be treated as Alamouti decoding, resulting in $2M$ branch-metric computations.

Moreover, if two hard-decision, symbol-by-symbol decoding steps in each branch of the 6-dimensional real SD are taken, the decoding complexity does not exceed $2M^6$.

VI. Conclusion

Motivated by the fast-decodable 2×2 STBC in [1, 3], we present a new family of full-rate STBC for 4×2 MIMO. First, using a combination of both algebraic and quasi-orthogonal STBC structures, we exhibit a new STBC in this family which outperforms any known code for 4×2 MIMO. Second, for the proposed STBC, we propose a reduced-complexity sphere decoding algorithm, which enables using only a 12-dimensional real SD, rather than the standard 16-dimensional SD.

References

[1] J. Paredes, A. B. Gershman, and M. G. Alkhanari, "A 2×2 space–time code with non-vanishing determinants and fast maximum likelihood decoding," in *Proc. ICASSP*, Honolulu, Hawaii, USA, April 2007.

[2] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE J. Select. Areas Commun.*, vol. 16, no. 8, pp. 1451–1458, October 1998.

[3] A. Hottinen and O. Tirkkonen, "Precoder designs for high rate space–time block codes," *Proc. CISS 2004*, Princeton, March 17-19, 2004.

[4] M. Samuel and M. P. Fitz, "Reducing the detection complexity by using 2×2 multi-strata space–time codes," *Proc. IEEE ISIT 2007*, pp. 1946–1950, Nice, France, June 2007.

[5] V. Tarokh, N. Seshadri and A. R. Calderbank, "Space–time codes for high data rate wireless communications: Performance criterion and code construction," *IEEE Trans. Inform. Theory*, vol. 44, no. 2, pp. 744–765, March 1998.

[6] J.-C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden Code: A 2×2 full-rate space–time code with non-vanishing determinants," *IEEE Trans. Inform. Theory*, vol. 51, no. 4, pp. 1432–1436, Apr. 2005.

[7] E. Viterbo and E. Biglieri, "A universal lattice decoder," in *GRETSI 14-eme Colloque*, Juan-les-Pins, France, Sept. 1993.

[8] E. Viterbo and J. Boutros, "A universal lattice code decoder for fading chanel," *IEEE Trans. Inform. Theory*, vol. 45, pp. 1639–1642, July 1999.

[9] C.-P. Schnorr and M. Euchner, "Lattice basis reduction: Improved practical algorithms and solving subset sum problems," *Mathematical Programming*, vol. 66, pp. 181–199, 1994.

[10] M. O. Damen, H. El Gamal, and G. Caire, "On maximum-likelihood detection and the search for the closest lattice point," *IEEE Trans. Inform. Theory*, vol. 49, no. 10, pp. 2389–2402, October 2003.

[11] H. Jafarkhani, "A quasi-orthogonal space–time block code," in *IEEE Trans. Commun.*, vol. 49, no. 1, pp. 1–4, January 2001.

[12] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, "Perfect space–time block codes," *IEEE Trans. Inform. Theory*, vol. 52, no. 9, pp. 3885–3902, September 2006.

[13] Y. Hong, E. Viterbo, and J.-C. Belfiore, "A space–time block coded multiuser MIMO downlink transmission scheme," *Proc. IEEE ISIT*, pp. 257–261, Seattle, WA, June–July 2006.