Adaptive Polar Coding with High Order Modulation for Block Fading Channels

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***Abstract*—We consider the design of polar codes for block fading channels. The key idea is to combine modulation, fading, and coding in a single entity. This design is based on two facts: (i) for each fading block, symbols with different fading coefficient has different reliability; (ii) for each symbol, different bit levels of a high order modulation observe different noise levels. In other words, the bit channels are *partially polarized* by modulation and fading. This new viewpoint inspires us to construct polar codes by matching code polarization perfectly with modulation polarization and fading polarization. The resulting codes adapt to the channel quality fluctuation, thus provide better performance than conventional polar BICM schemes and LDPC codes.**

I. Introduction

The changing signal-to-noise ratio (SNR) exhibited by a fading channel can cause severe error bursts. A classical method for combating the channel quality fluctuation is adaptive coded modulation (ACM), utilizing a feedback channel to provide channel state information at the transmitter (CSIT) [1–3]. In ACM, the channel coding and modulation designs are separable. The error bursts are eliminated by adjusting the size of the transmitted constellation according to the quality of the channel. However, this technique suffers from a varying system throughput, making it unsuitable when fixed throughput is required. Deriving an adaptive transmission scheme with fixed constellation size and data rate is particularly interesting.

One promising research direction is to use CSIT in the design of codes for fading channels. In general, it is impractical to adapt an arbitrary code to each channel state, but particular structure of polar codes [4], which making this approach feasible. The idea of polar code is to transform a communication channel into polarized subchannels: either completely noisy or noiseless. Information bits are then transmitted over the noiseless subchannels, while fixed or *frozen bits* are send over the noisy ones. Polar codes are particularly suited to fading channels, as they adapt to varying channel quality. The low complexity of polar code construction allows the implementation of a real-time adaptive coding. In practice, polar codes can be firstly designed at the receiver side according to CSI, and then the frozen indices are fed back to the transmitter.

Recently, many efforts have been made to construct polar-based schemes for transmissions over fading channels. In [5], polar lattices for fading channels are constructed. In [6], the author models the subchannels induced by the polarizing transformation as multi-path fading channels and tracks their diversity order and noise variance. In [7], an embedded polar coding scheme is proposed for fading binary symmetric channels. In [8], the author constructs polar codes for block Rayleigh fading channels under the assumption that only channel statistics are available. In [9], the authors study the polarization of block fading channel with two distinct fading coefficients. However, the methods in [6–9] are limited to bipolar signaling, i.e., transmitting one bit per channel use. It remains an open problem to construct polar codes with high order modulation for fading channels.

This work is supported by the Australian Research Council Discovery Project with ARC DP160100528.

In this paper, we propose an explicit method to construct polar codes for block fading channels with arbitrary input alphabet size. We note that the reliability of each transmitted symbol varies with its fading coefficient. Moreover, the bits with different modulation level observe different noise level. In analogy to code polarization, these effects can be viewed as *fading polarization* and *modulation polarization*. This new viewpoint allows us to combine coding, fading, and modulation in a single entity for improved performance. In other words, the proposed polar coding scheme matches code polarization perfectly with the fading polarization and the modulation polarization, and thus provides better performance than conventional polar or LDPC BICM [10,11].

Section II presents system model. Section III describes the proposed construction of polar codes. Section IV shows simulation results and comparisons with other codes. Section V sets out both theoretical and practical conclusions. The Appendix contains the proof of the theorem.

II. System Model and Problem Statement

A. System Model

We consider the discrete-time channel model

$$y_k = h_k s_k + n_k, \quad k = 1, ..., L, \tag{1}$$

where L is the frame size. In the k^{th} channel use, s_k is the transmitted symbol in an M-PAM constellation $\mathcal{P}_M$:

$$\mathcal{P}_M = \{-M+1, -M+3, ..., M-1\}, \tag{2}$$

y_k is the channel output, n_k is a zero mean Gaussian noise, $n_k \sim \mathcal{N}\left(0, \sigma^2\right)$, and h_k is the channel gain. Here, h_k and

n_k are assumed to be independent. We do not specify the fading type or the distribution of h_k, since we assume that the transmitter knows the realization of h_k.

In the *block fading* channel model, a transmission frame of L symbols is affected by $1 \leq B \leq L$ independent fading realizations, resulting in a block of L/B symbols being affected by the same fading realization. Different values B represents different types of fading, e.g., for $B = L$, we refer to *fast fading* and for $B = 1$ to *slow fading*.

We assume that each transmitted frame contains one codeword, i.e., in each frame, a rate-R encoder maps K information bits $\{u_i\}_1^K$ into N coded bits $\{x_i\}_1^N$, where

$$N = L \log_2(M). \tag{3}$$

Let $R \triangleq K/N$ be the code rate. After then, every $\log_2(M)$ coded bits are modulated to generate a signal using M-PAM modulation in (2) with Gray mapping.

In this work, we fix the code rate to $R = 1/2$. As shown in [9], the *mutual information outage* (MIO) occurs if

$$\frac{1}{L}\sum_{k=1}^{L} I(s_k; y_k) < R = \frac{1}{2}. \tag{4}$$

To avoid MIO, transmission control can be employed, i.e., if MIO occurs, the transmitter will not send messages.

B. Polar Codes

1) Code Construction: Polar codes are constructed based upon a phenomenon called channel polarization discovered by Arıkan [4]. The polarization matrix is given as

$$\mathbf{G} = \mathbf{B}_N \mathbf{F}^{\otimes n}, \tag{5}$$

where $N = 2^n$, $\mathbf{B}_N$ is a bit-reversal permutation matrix [4],

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \tag{6}$$

and $(\cdot)^{\otimes n}$ denotes the n fold Kronecker product of a matrix recursively defined by $\mathbf{F}^{\otimes n} = \mathbf{F} \otimes \mathbf{F}^{\otimes (n-1)}$.

Consider N independent copies of a binary-input discrete memoryless channel $W : \{0, 1\} \to \mathcal{Y}$, where $\mathcal{Y}$ is output alphabets. The N input bits $\mathbf{u} = [u_1, u_2, ..., u_N]$, which are selected from uniform distributions, are multiplied by $\mathbf{G}$ in (5) and then transmitted over the N copied of W. Let $\mathbf{y} = [y_1, y_2, ..., y_N]$ be the channel output bits. Assuming the past bits are decoded successfully and known, the *bit-channel* observed by u_i can be written as

$$W^{(i)} \triangleq W^{(i)}(\mathbf{y}, \{u_t\}_1^{i-1} | u_i), \quad i = 1, ..., N. \tag{7}$$

Channel polarization is the fact that the capacity of $W^{(i)}$, denoted as $I(W^{(i)})$, approached to either 1 or 0, as N goes to infinity. These polarized channels $\left\{W^{(i)}\right\}_1^N$ suggest a channel coding scheme, referred to as polar codes, by transmitting a sequence of K information bits over K noiseless channels and transmitting a sequence of $N-K$ fixed or *frozen bits* over $N-K$ noisy channels. Different form conventional linear block codes, the main challenge for polar code design is finding $\mathcal{F}$.

The Bhattacharyya parameter, denoted as $Z(W)$, can be used to measure the error probability of a channel W. It is the upper bound of error probability under maximum-likelihood (ML) decoding and is defined as [4]

$$Z(W) = \sum_{y \in \mathcal{Y}} \sqrt{p(y|0)p(y|1)}. \tag{8}$$

Let $Z^{(i)}$, $i = 1, ..., N$, represent the Bhattacharyya parameter of the bit channel $W^{(i)}$ in (7). The indices of the $N-K$ largest values in the set $\left\{Z^{(i)}, i = 1, ..., N\right\}$ form the frozen bit set $\mathcal{F}$. Let $Z_0^{(i)} = Z(W)$ and $Z_n^{(i)} = Z^{(i)}$. For binary erasure channel (BEC), $Z_n^{(i)}$ can be computed recursively from $Z_0^{(1)}$ as [4]

$$Z_{j+1}^{(i)} = \begin{cases} 2Z_j^{(i)} - \left(Z_j^{(i)}\right)^2, & 1 \leq i < 2^j + 1 \\ \left(Z_j^{(i-2^j)}\right)^2, & 2^j + 1 \leq i \leq 2^{j+1} \end{cases} \tag{9}$$

2) Polar Encoding: The encoder of a polar code is

$$\mathbf{x} = \mathbf{u}\mathbf{G} = \mathbf{u}\mathbf{B}_N\mathbf{F}^{\otimes n}. \tag{10}$$

The the input bit sequence $\{u_i\}_1^N$ are divided into two sets: the information bits set $\mathcal{A}$ and the frozen bit set $\mathcal{F}$. The frozen bits are not decoded given that they are known a priori at receiver.

3) Polar Decoding: Arikan proposes the *successive cancelation decoding* (SC) procedure summarized in Algorithm 1. In line 5 of Algorithm 1, $W^{(i)}(\mathbf{y}, \{\hat{u}_t\}_1^{i-1} | u_i)$ represents the likelihood of u_i given the channel output $\mathbf{y}$ and the previously decoded bits $\{\hat{u}_t\}_1^{i-1}$.

Algorithm 1 Successive Cancelation Decoding [4]

1: **for** $i = 1, 2, ..., N$ **do**
2:     **if** $i \in \mathcal{F}$ **then**
3:         $\hat{u}_i \leftarrow u_i$;   //this is a frozen bit
4:     **else**
5:         $\hat{u}_i \leftarrow \arg\max_{u_i \in \{0,1\}} W^{(i)}(\mathbf{y}, \{\hat{u}_t\}_1^{i-1} | u_i)$;
6:     **end if**
7: **end for**

In the SC decoder, past decisions are never revisited in the future. In return for this sub-optimality, the likelihoods $W^{(i)}(\mathbf{y}, \{\hat{u}_t\}_1^{i-1} | u_i)$ can be computed efficiently and the decoding complexity scales like $O(N \log N)$. To enhance the decoder performance, the list SC decoder was introduced in [12]. The idea is to keep a list of most likely codewords at each decoding step. When the last codeword bit has been decoded, the most likely codeword in the list is returned as the decoded codeword. For small block length, the performance of list decoder is very close to the ML decoder.

C. Problem Statement

With the received signal being scaled by the channel gain, the block fading channel in (1) can be treated as a set of independent AWGN channels with different received SNRs.

$$\hat{y}_k = s_k + \hat{n}_k, \quad k = 1, ..., L, \tag{11}$$

where $\hat{y}_k = y_k/h_k$, $\hat{n}_k \sim \mathcal{N}\left(0, \sigma_k^2\right)$, and $\sigma_k^2 = \sigma^2/h_k^2$.

1) Fading Polarization: The symbol error probability of the k^{th} channel in (11) is

$$P_{\mathrm{s}}(\hat{s}_k \neq s_k) = 2\frac{M-1}{M}Q\left(\frac{|h_k|}{\sigma}\right), \tag{12}$$

where $Q(\cdot)$ is the complementary error function. The channels with large fading coefficients are more reliable than the ones with small fading coefficients. In other words, the channels are partially "polarized" by the fading process. In analogy to code polarization, we refer to this effect as *fading polarization.*

2) Modulation Polarization: Since the bits $\{x_i\}_{(k-1)m+1}^{km}$, where $m = \log_2 M$, are mapped to the symbol s_k, under Gray mapping, the bit error probability of x_j can be easily computed as

$$P_{\mathrm{b}}(\hat{x}_j \neq x_j) \approx \frac{1}{2^{m-(j \bmod m)}}Q\left(\frac{|h_k|}{\sigma}\right), j = (k-1)m+1, ..., km \tag{13}$$

The approximation in (13) ignores high order terms and is quite accurate in practice. We observe that the bits with different indexes have different reliability. In other words, the bits are partially "polarized" by the modulation operation. We refer to this effect as *modulation polarization.*

In this work, we take advantage of fading and modulation polarization to construct polar codes. We will optimize the mapping of the coded bits to the fading coefficients and modulation levels, so that the frame error probability is minimized.

III. Construction of Polar Codes for Block Fading Channels with High Order Modulation

We first introduce our design criterion and then derive the optimal mapping between coded bits and fading coefficients.

A. Design Criterion

We consider the mapping between the coded bits $\{x_i\}_1^N$ and the AWGN channels in (11). Without loss of generality, let $\left\{x_{\rho(i)}\right\}_1^N$ be a permuted version of $\{x_i\}_1^N$ by an index permutation. The received signals in (11) can be written as

$$\hat{y}_k = \mathcal{M}\left(\left\{x_{\rho(k)}\right\}_{(k-1)m+1}^{km}\right) + \hat{n}_k, \quad k = 1, ..., L, \tag{14}$$

where $\mathcal{M}(\cdot)$ denotes the PAM modulation operation.

To gain more intuition, we denote the pairs of coded bit, modulation level, and channel SNR as

$$\mathcal{L}_\rho \triangleq \left\{\left(x_{\rho(i)}; \eta_i, \frac{\hat{h}_i^2}{\sigma^2}\right)\right\}_1^N, \tag{15}$$

where

$$\eta_i = i \bmod m, \; \hat{h}_i = h_{\lceil i/m \rceil}. \tag{16}$$

The operation $\lceil x \rceil$ rounds x to the next larger integer. Note that η_i represents the i^{th} coded bit index in the modulation symbol label, while $\hat{h}_i$ represents the fading for the i^{th} coded bit. Together, the pair $\left(\eta_i, \hat{h}_i^2/\sigma^2\right)$ represents the i^{th} *coded bit channel* after bit labeling and fading.

With a different permutation ρ, a coded bit can be assigned to a different modulation level and a different channel. To simplify notation in the code design, we prefer to use the natural order of coded bits. The pairing $\mathcal{L}$ in (15) can be rewritten as

$$\mathcal{L}_\rho = \left\{\left(x_i; \eta_{\bar{\rho}(i)}, \frac{\hat{h}_{\bar{\rho}(i)}^2}{\sigma^2}\right)\right\}_1^N, \tag{17}$$

where $\bar{\rho}$ is the inverse permutation of ρ.

Remark 1: From (17), the modulation level and channel index of a coded bit is uniquely determined by permutation ρ.

Since the construction of polar codes depends on the pairing $\mathcal{L}_\rho$, different permutations lead to different performance. Our design criterion is to find the optimal permutation ρ maximizing the performance of polar codes.

For a given permutation $\bar{\rho}$, let $W^{(i)} : u_i \to \hat{\mathbf{y}}$ where $\hat{\mathbf{y}} = [\hat{y}_1, \hat{y}_2, ..., \hat{y}_L]$, denotes the i^{th} bit channel, where

$$W^{(i)} \triangleq W^{(i)}(\hat{\mathbf{y}}, \{u_t\}_1^{i-1} | u_i). \tag{18}$$

Let $Z_n^{(i)}$ be the Bhattacharyya parameter of $W^{(i)}$. The upper bound on the block error probability can be calculated as

$$P_{\mathrm{B}} = \sum_{i \in \mathcal{A}} Z_n^{(i)}, \tag{19}$$

where $\mathcal{A}$ is the information set corresponding to K smallest Bhattacharyya parameters. Our code design criterion will be

$$\bar{\rho}_{\mathrm{opt}} = \arg\min_{\bar{\rho}} P_{\mathrm{B}}. \tag{20}$$

B. Computing $Z_n^{(i)}$

Here we show how to compute $Z_n^{(i)}$ in (20) for a given $\bar{\rho}$. Recalling that the polar encoder contains $n = \log_2 N$ stages. In each stage, the encoder first pairs the N bit channels in a predefined order, and then polarizes each pair. Fig. 1 demonstrates the operation of channel polarization: two independent binary channels W_1 and W_2 are selected and transformed to a pair of binary channels W' and W''.

Let $W_j^{(i)}$ denote the i^{th} bit channel in stage j, and $Z_j^{(i)}$ be the Bhattacharyya parameter of $W_1^{(i)}$, where $j = 0, 1, ..., n$. Note that $j = 0$ represents the initial stage, i.e., when $j = 0$, $W_0^{(i)}$ represents the bit channel $x_i \to y^{(i)}$:

$$y^{(i)} = \mathcal{M}\left(\{b_t\}_1^m\right) + n^{(i)}, \tag{21}$$

where $n^{(i)} \sim \mathcal{N}\left(0, \sigma^2/\hat{h}_i^2\right)$ and $b_{\eta_{\bar{\rho}(i)}} = x_i$.

From (21), the value of $Z_0^{(i)}$ can be calculated by [13]

$$Z_0^{(i)} = \int_{-\infty}^{+\infty} \sqrt{p(y^{(i)}|b_{\eta_{\bar{\rho}(i)}} = 0)p(y^{(i)}|b_{\eta_{\bar{\rho}(i)}} = 1)}dy^{(i)} \tag{22}$$

Note that the integral in (22) can be written in the form

$$\int_{-\infty}^{+\infty} e^{-x^2} f(x)dx. \tag{23}$$

It can be computed efficiently using *Gauss–Hermite quadrature* [14].

Note that the value of $Z_0^{(i)}$ varies with the index i, thus is not a constant. It means that in the future encoding stages, we need to polarize bit channels with different reliability. This is different from the standard polar codes design in [4]

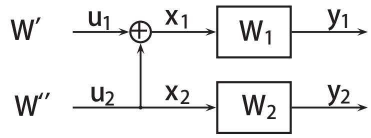

Fig. 1. One level of channel polarization: $(W_1, W_2) \longmapsto (W', W'')$

[13], where channels with the same reliability are polarized. Therefore, the recursion in (9) can not be applied.

In summary, the difficulty of computing $Z_n^{(i)}$ from $Z_0^{(i)}$ lies in polarizing bit channels with different Bhattacharyya parameters. This problem is solved by the following lemma.

Lemma 1: Suppose that $(W_1, W_2) \longmapsto (W', W'')$ for some set of binary-input channels (Fig. 1). Then,

$$Z(W'') = Z(W_1)Z(W_2), \tag{24}$$

$$Z(W') \leq Z(W_1) + Z(W_2) - Z(W_1)Z(W_2), \tag{25}$$

$$Z(W') \geq Z(W_1) \geq Z(W''). \tag{26}$$

Equality holds in (25) iff W_1 and W_2 are BECs. We have $Z(W') = Z(W'')$ iff $Z(W_1) = Z(W_2) = 0$ or 1.

Proof: The proof is similar to that of [4, Proposition 5]. ■

Adding (24) and (25) in Lemma 1 shows that the reliability can only improve after channel polarization in the sense that

$$Z(W') + Z(W'') \leq Z(W_1) + Z(W_2), \tag{27}$$

with equality iff W_1 and W_2 are BECs.

In this work, we use (24) and the upper bound (25) to actually estimate the reliability of each bit channels. Starting from $Z_0^{(i)}$ in (22), we can recursively compute the upper bounds on $Z_n^{(i)}$ using (24) and (25). The frozen bits are then selected based on these upper bounds. This method does not require the exact value of Bhattacharyya parameters, thus greatly simplifies the construction of polar code.

C. Finding $\bar{\rho}_{opt}$

The number of possible permutation patterns in (20) is $N!/((N/B)!)^B$, where B is the number of distinct fading coefficients. For large N, it is impossible to solve (20) by an exhaustive search. To reduce the search space, we relax the optimization problem in (20) to

$$\bar{\rho}_{\text{opt},1} = \arg\min_{\bar{\rho}} \sum_{i=2,4,\ldots,N} Z_1^{(i)}. \tag{28}$$

In other words, we minimize the block error probability of the even-indexed channels $W_1^{(i)}$, $i = 2, 4, ..., N$, in the first polarization stage. The motivation behind this choice comes from (26), i.e., after each polarization operation, the even-indexed channel becomes more reliable than the odd-indexed one. Optimizing the performance of even-indexed channels in the first stage will create a good starting point for the whole polarization process. Simulations verified that the relaxation in (28) serves as a good criterion in practice.

We consider the *un-permuted* pairs in (17), i.e., $\bar{\rho}(i) = i$

$$\mathcal{L}_\phi \triangleq \left\{ \left(x_i; \eta_i, \frac{\hat{h}_i^2}{\sigma^2} \right) \right\}_1^N. \tag{29}$$

Consequently, the Bhattacharyya parameter in (22) reduces to

$$\hat{Z}_0^{(i)} = \int_{-\infty}^{+\infty} \sqrt{p(y^{(i)}|b_{\eta_i} = 0)p(y^{(i)}|b_{\eta_i} = 1)}dy^{(i)}. \tag{30}$$

This problem in (28) is solved by the following theorem.

Theorem 1: Let $\left\{ \hat{Z}_0^{(\kappa(i))} \right\}_1^N$ be a sorted version of $\left\{ \hat{Z}_0^{(i)} \right\}_1^N$ in descending order, i.e., $\hat{Z}_0^{(\kappa(i))} \geq \hat{Z}_0^{(\kappa(j))}$ if $i \leq j$. The solution of (28) is

$$\bar{\rho}_{\text{opt},1} = \pi(\Phi), \tag{31}$$

for any arbitrary permutation π, where Φ is the set of pairs of indices in the set $\{\kappa(i)\}_1^N$, defined by

$$\begin{aligned} \Phi &= \{(\Phi(i,1), \Phi(i,2))\}_1^{N/2} \\ &= \{(\kappa(1), \kappa(N)), (\kappa(2), \kappa(N-1)), \cdots, (\kappa(N/2), \kappa(N/2+1))\} \end{aligned} \tag{32}$$

i.e., pairing the index of the largest Bhattacharyya parameter with the smallest.

Proof: See Appendix A. ■

Theorem 1 shows that the optimal permutation is unique up to a permutation π. In practice, we can always use $\bar{\rho}_{\text{opt},1} = \Phi$ and the pairing between coded bits, modulation level and channel index can be written as

$$\begin{aligned} \mathcal{L}_{\text{opt}} = \Bigg\{ & \left(x_1; \eta_{\kappa(1)}, \frac{\hat{h}_{\kappa(1)}^2}{\sigma^2} \right), \left(x_2; \eta_{\kappa(N)}, \frac{\hat{h}_{\kappa(N)}^2}{\sigma^2} \right), \cdots, \\ & \left(x_{N-1}; \eta_{\kappa(N/2)}, \frac{\hat{h}_{\kappa(N/2)}^2}{\sigma^2} \right), \left(x_N; \eta_{\kappa(N/2+1)}, \frac{\hat{h}_{\kappa(N/2+1)}^2}{\sigma^2} \right) \Bigg\} \end{aligned} \tag{33}$$

Example 1: In Fig. 2, we demonstrate the construction of polar codes with $N = 8$. In stage 0, we first compute $\left\{ \hat{Z}_0^{(i)} \right\}_1^N$ according to (30). After then, we sort $\left\{ \hat{Z}_0^{(i)} \right\}_1^N$ to obtain the permutation order κ. Now we are able to map the coded bits to the modulation levels and channels:

- If i is odd, we map x_i to the modulation level $\eta_{\kappa((i+1)/2)}$ in the channel with fading coefficient $\hat{h}_{\kappa((i+1)/2)}$.
- If i is even, we map x_i to the modulation level $\eta_{\kappa(N+1-i/2)}$ in the channel with fading coefficient $\hat{h}_{\kappa(N+1-i/2)}$.

Finally, we compute

$$Z_0^{(i)} = \begin{cases} \hat{Z}_0^{(\kappa((i+1)/2))}, & \text{if } i \text{ is odd} \\ \hat{Z}_0^{(N+1-i/2)}, & \text{otherwise} \end{cases} \tag{34}$$

From stage 1 to 3, we compute successively $Z_1^{(i)}$, $Z_2^{(i)}$, and $Z_3^{(i)}$ by using (34), (24) and (25). We select the frozen bits according to $Z_3^{(i)}$.

In summary, the proposed method allows us to design polar codes in real time and track the changes in channels.

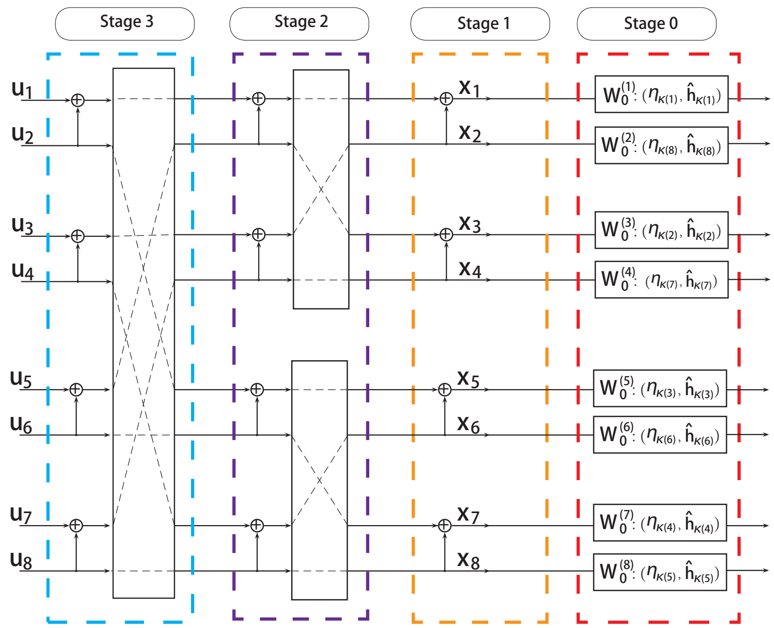

Fig. 2. Construction of polar codes with mapping Φ in (32): If i is odd, x_i is mapped to the modulation level $\eta_{\kappa((i+1)/2)}$ in the channel with fading coefficient $\hat{h}_{\kappa((i+1)/2)}$; If i is even, x_i is mapped to the modulation level $\eta_{\kappa(N+1-i/2)}$ in the channel with fading coefficient $\hat{h}_{\kappa(N+1-i/2)}$.

IV. Simulation Results

This section examines the performance of the proposed polar codes. We consider Rayleigh fading channel, i.e.,

$$h_i \sim 2h_i \exp(-h_i^2). \tag{35}$$

The frame size is 512. In each fame, there are 2 distinct fading coefficients, denoted as, $\{h_1, h_2\}$, i.e., every block of length 256 symbols will be affected by the same fading coefficient. To avoid mutual information outage in (4), we apply a simple transmission control protocol, i.e., the transmitter will not send message if

$$\max\{h_1, h_2\} < T, \tag{36}$$

where T is a threshold. This outage occurs with probability

$$P_{\text{out}} \triangleq \Pr\left(\max\{h_1, h_2\} < T\right) = \Pr\left(h_1 < T\right)\Pr\left(h_2 < T\right). \tag{37}$$

We set $P_{\text{out}} = 0.0489$ with $T = 0.5$, i.e., for every 100 transmission frames, on average 5 frames are not suitable for transmission. We construct $(512, 256)$ polar codes using the mapping proposed in (33). For comparison purposes, the performance of $(512, 256)$ polar codes with BICM, and $(512, 256)$ LDPC codes from IEEE 802.11ad are also shown. In the polar BICM, the coded bits are random interleaved and then modulated. Since the interleaved bit channels are approximately equally likely, we construct the codes by setting

$$Z_0^{(i)} = \frac{1}{N}\sum_{t=1}^{N} \hat{Z}_0^{(t)}, \quad i = 1, ..., N. \tag{38}$$

Fig. 3 illustrates the block error rate of proposed polar codes under SC decoding with 4-PAM. For the LDPC codes, belief propagation decoding with up to 50 iterations was used. The proposed polar codes provide significant gain with respect to LDPC. The reason is that LDPC cannot adapt to the channel changes. We find that the polar BICM perform quite badly over fading channels. This phenomenon has also been observed in [6]. The loss is due to the mismatch between the code polarization and fading polarization. This result confirms that the performance of polar codes is dominated by the mapping between coded bits, modulation level, and channels.

Fig. 4 shows the performance of proposed polar codes with 16-PAM. We observe similar tendencies, i.e., the proposed polar codes outperform polar BICM and LDPC. The SNR gap to LDPC is 8 dB at block error rate 10^{-4}. This result confirms that our construction is valid for high order modulation.

V. Conclusions

In this paper, a method for constructing polar codes for fading channels with high order modulation was proposed. Different from traditional adaptive coded modulation approaches, we adapt the code structure to the channel changes, given fixed constellation size and code rate. The novelty of our design is to optimize the mapping of the coded bits to the modulation levels and fading channels. The simulation results show that with our design, polar codes provide 8 dB gain with respect to LDPC codes at block error rate 10^{-4} for 16-PAM.

Appendix

A. Proof of Theorem 1

For an arbitrarily permutation $\bar{\rho}$ in (17), we have

$$Z_0^{(i)} = \hat{Z}_0^{(\bar{\rho}(i))}, \quad i = 1, ..., N \tag{39}$$

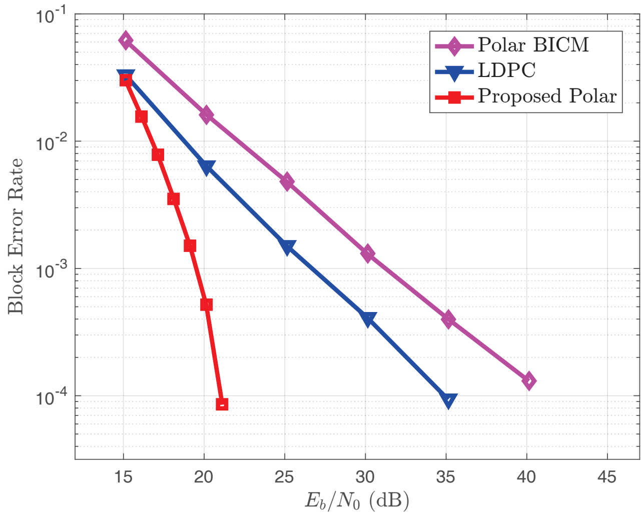

Fig. 3. Performance of different rate-$1/2$ codes in Rayleigh fading channels with 4-PAM

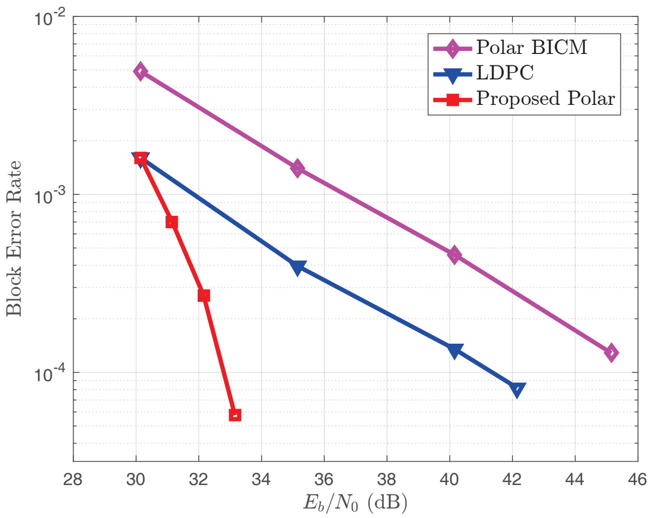

Fig. 4. Performance of different rate-$1/2$ codes in Rayleigh fading channels with 16-PAM

where $\hat{Z}_0$ is given in (30). From (24), we have

$$\sum_{i=2,4,\ldots,N} Z_1^{(i)} = \sum_{i=2,4,\ldots,N} Z_0^{(i-1)} Z_0^{(i)} = \sum_{i=2,4,\ldots,N} Z_0^{(\bar{\rho}(i-1))} Z_0^{(\bar{\rho}(i))}. \quad (40)$$

For simplicity, let

$$\theta(\bar{\rho}(i)) = \hat{Z}_0^{(\bar{\rho}(i))}. \quad (41)$$

We derive the relation

$$\begin{aligned} & \sum_{i=2,4,\ldots,N} \theta(\bar{\rho}(i-1))\theta(\bar{\rho}(i)) \\ = \; & \frac{1}{2} \sum_{i=2,4,\ldots,N} [(\theta(\bar{\rho}(i-1))\theta(\bar{\rho}(i)) + \theta(\bar{\rho}(i))\theta(\bar{\rho}(i-1)))] \\ = \; & \frac{1}{2} \sum_{t=1,2,\ldots,N} \theta(t)\theta(\mathcal{P}(t)), \end{aligned} \quad (42)$$

where the sequence $\{\mathcal{P}(t)\}_1^N$ represents a permutation of $\{t\}_1^N$.
Recalling that

$$\theta(\kappa(1)) \geq \theta(\kappa(2)) \geq \cdots \geq \theta(\kappa(N)). \quad (43)$$

According to the *rearrangement inequality* in [15], for any permutation $\mathcal{P}$, it holds

$$\begin{aligned} & \frac{1}{2} \sum_{t=1,2,\ldots,N} \theta(t)\theta(\mathcal{P}(t)) \\ \geq \; & \frac{1}{2} \sum_{t=1,2,\ldots,N} \theta(\kappa(t))\theta(\kappa(N-t+1)) \\ = \; & \sum_{t=1,2,\ldots,N/2} \theta(\kappa(t))\theta(\kappa(N-t+1)). \end{aligned} \quad (44)$$

From (42) and (44), for any permutation ρ, we have

$$\sum_{i=2,4,\ldots,N} Z_1^{(i)} \geq \sum_{t=1,2,\ldots,N/2} \theta(\kappa(t))\theta(\kappa(N-t+1)). \quad (45)$$

The equality holds when

$$\bar{\rho} = \pi(\Phi), \quad (46)$$

which yields (31).

■

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