Trellis Coded Modulation for Informed Receivers

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***Abstract*—We consider the following variant of the index coding problem. A transmitter wishes to communicate several messages, each of which is k bit long, to multiple *informed receivers* over a noisy broadcast link. Each informed receiver already knows some of the messages a priori, which is called its *side information*, and demands to know the rest of the messages. However, unlike in an index coding problem, the transmitter is oblivious of the side information available at the receivers. We propose a trellis coded modulation (TCM) based scheme, called *TCM for informed receivers* (TCMIR), for this scenario. The transmitter jointly encodes the messages using a $^k/_{k+1}$ convolutional code and transmits the $k+1$ coded bits as a signal from a 2^{k+1} point constellation. New bit labeling rules are given to guarantee that the performance gain of any receiver is independent of the actual bit value of the side information. Compared to the uncoded case, each receiver can decode the required messages more reliably. Moreover, the receivers with side information can recover the remaining messages more reliably compared to a receiver with no side information. Next, the performance of TCMIR is compared with that of nested codes available in the literature and it is shown that TCMIR outperforms the nested codes for a receiver with side information.**

I. Introduction

The fifth generation cellular network architecture (5G) is intended to offer improved data rate, reduced latency, and better quality of service compared to 4G and huge device connectivity [1]. With improved network connectivity, it is expected that several tens of billions of devices will operate in near future. Consider the scenario of a transmitter communicating several messages (like popular multimedia content, software updates, or daily news and weather report) to multiple clients over a noisy broadcast link. The messages are first broadcast one at time but the receivers, possibly because of poor channel condition, may be able to recover only some of the messages which are then stored in their caches. In an *index coding* problem [2], the transmitter is aware of the client caches (the receivers send acknowledgment to the transmitter for each message successfully recovered) and then broadcasts coded messages, called an index code, that ensure successful recovery of each client's demanded messages; this increases the transmission rate. But sending acknowledgment requires network resources, viz., bandwidth and energy, and may increase latency.

We consider the case where the transmitter will be unaware of the client caches. But it can still leverage the knowledge that some clients may have successfully recovered some of the messages to reduce error rate by broadcasting coded messages. This problem is referred to as *coding for informed decoders* in [3]. Multi-terminal coding and decoding schemes are key features of relay network and cooperative diversity. In these scenarios, the broadcast signal can be overheard by multiple receivers. Coding schemes that utilize these overheard signals at receivers as side information have been proposed to decode the required messages [4]–[7]. These schemes are based on error correcting codes and enable the receivers with side information to achieve gains in transmission power efficiency. Binary convolutional codes for this problem, called *nested codes*, were studied in [4], [5]. A nested code with 4 messages to be transmitted was proposed in [6] for wireless relay network. Bit labeling schemes for multidimensional QAM that can utilize side information at the signal demodulator were given in [7].

In this paper, we study the problem of communicating multiple messages to multiple clients over a noisy broadcast link. Each client already knows some of the messages, called its side information, and demands to know the rest. Oblivious of the clients' side information, the objective of code design at the transmitter is to exploit the side information at each client to minimize the bit error rate. In Section II, we present the system model. The contributions of this paper are as follows:

1) In Section III, we propose a trellis coded modulation (TCM) based scheme, called TCM for informed receivers (TCMIR).
2) In Section III-C, we give new bit labeling rules that ensures that the coding gain afforded by the side information of a receiver is independent of the actual bit value of the side information. We also illustrate with examples that if these rules are not satisfied a receiver with side information may not get any coding gain.
3) In Section IV, performance analysis of TCMIR based on d_{free}. For the case in which the transmitter wants to communicate 2 messages, we perform code search for good TCMIR schemes; these are listed in the appendix.

In Section V, simulation results that compare the nested codes with TCMIR is given. Section VI concludes the paper.

II. System Model

The system model is illustrated in Fig. 1. A source wants to communicate L messages $\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_L$ to multiple receivers, and each message contains K bits, i.e., $\mathbf{m}_i \in \{0,1\}^K$. Some receivers already have a subset of the L messages. Let S_j denote the set of message indices already known to the jth receiver. The realization of $\mathbf{m}_l$ is known for all $l \in \mathsf{S}_j$ as side information, i.e., $\mathbf{m}_l = \mathbf{a}_l$ for $l \in \mathsf{S}_j$.

The L messages are first encoded by a rate $^k/_n$ channel encoder ρ and the output is a single codeword $\mathbf{c}$ with N bits, i.e., $\rho : (\mathbf{m}_1, \ldots, \mathbf{m}_L) \mapsto \mathbf{c} \in \{0,1\}^N$. Let $\mathscr{C}$ denote the codebook of all 2^{KL} possible codewords. The binary sequence $\mathbf{c}$ is then mapped to a sequence $\mathbf{v}$ of JL two-dimensional modulation symbols from a 2^n-ary constellation

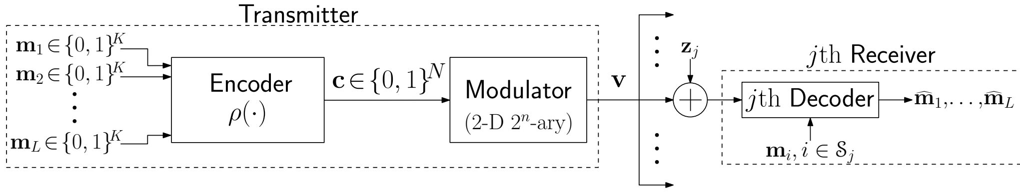

Fig. 1. The system model of the transmitter broadcasting L messages. The jth receiver knows the realization of $\mathbf{m}_i, i \in \mathsf{S}_j$ as side information.

(relation among integers k, n, K, L, and J will be specified in Section III). The spectral efficiency is given by $\frac{KL}{2N}n$ bits per dimension (b/dim). Then, $\mathbf{v}$ is broadcast to each receiver over an *additive white Gaussian noise* (AWGN) channel. The received sequence at the jth receiver is $\mathbf{r}_j = \mathbf{v} + \mathbf{z}_j$, where $\mathbf{z}_j$ is noise.

Given one message side information $\mathbf{m}_i = \mathbf{a}_i$ at a receiver, maximum likelihood decoding of $\mathbf{r}_j$ is achieved by decoding the size $2^{K(L-1)}$subcode $\mathscr{C}_{\mathbf{a}_i}$ of $\mathscr{C}$, where

$$\mathscr{C}_{\mathbf{a}_i} = \{\mathbf{c} = \rho(\mathbf{m}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{m}_L), \mathbf{m}_l \in \{0,1\}^K, l \neq i\}.$$

If the jth receiver knows all the messages with indices $l \in \mathsf{S}_j$, then it will decode the subcode $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}} = \bigcap_{l \in \mathsf{S}_j} \mathscr{C}_{\mathbf{a}_l} \subset \mathscr{C}$. Let d_0 and $d_{\mathbf{a}_{\mathsf{S}_j}}$ denote the minimum Euclidean distances of $\mathscr{C}$ and $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}$ respectively. Note that d_0 is also the minimum Euclidean distance of the subcode corresponding to no side information. The asymptotic coding gain due to side information is determined by the ratio $d^2_{\mathbf{a}_{\mathsf{S}_j}}/d_0^2$. Hence, the objective is to maximize both d_0 and the ratio $d^2_{\mathbf{a}_{\mathsf{S}_j}}/d_0^2$ for all $\mathsf{S}_j \subset \{1, 2, \ldots, L\}$. Note that in the case of a binary input binary output channel, the Hamming distance replaces the Euclidean distance.

III. TCM FOR INFORMED RECEIVERS

We now present the TCMIR scheme which involves joint design of an encoder and a signal mapper (modulator). The output of the encoder is mapped to a point in a finite constellation.

A. The Channel Encoder of TCMIR

In this section, we propose the encoder $\rho(\cdot)$ of TCMIR. The encoder is the concatenation of a selector and a linear convolutional encoder. We will show that the concatenated encoder preserves the linearity of each subcode.

Assume L messages $\{\mathbf{m}_1, \mathbf{m}_2, \ldots, \mathbf{m}_L\}$ are broadcast to multiple receivers, where each $K = Jk$ bits message is $\mathbf{m}_i = (b_1^i, b_2^i, \ldots, b_K^i)$, for some integer J.

Let $K = Jk$ and $\mathbf{m}_i = (b_1^i, b_2^i, \ldots, b_K^i)$ for $1 \leqslant i \leqslant K$. In our scheme, a selector Θ cyclically feeds k bits of each one of the L messages to a convolutional encoder. For example, for $K = 4$, $L = 2$, and $k = 2$, the 8 bits fed to the convolutional encoder are $(b_1^1, b_2^1, b_1^2, b_2^2, b_3^1, b_4^1, b_3^2, b_4^2)$. The overall KL bits $\mathbf{m}$ generated by the selector can be written as

$$\begin{aligned}\mathbf{m} &= \Theta(\mathbf{m}_1, \underbrace{0, 0, \ldots}_{L-1}) + \Theta(0, \mathbf{m}_2, \underbrace{0, \ldots}_{L-2}) + \ldots + \Theta(\underbrace{0, \ldots, 0}_{L-1}, \mathbf{m}_L) \\ &= \widetilde{\mathbf{m}}_1 + \widetilde{\mathbf{m}}_2 + \ldots + \widetilde{\mathbf{m}}_L,\end{aligned}$$

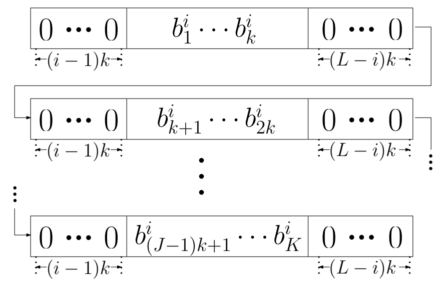

Fig. 2. The output $\widetilde{m}_i$ of selector Θ.

where $\widetilde{\mathbf{m}}_i$ is defined in Fig. 2. Codewords $\mathbf{c}$ are then generated by encoding $\mathbf{m}$ via a rate k/n convolutional encoder E_{cv}. Thus, the encoder of TCMIR is $\rho(\mathbf{m}_1, \ldots, \mathbf{m}_L) = E_{\text{cv}}[\Theta(\mathbf{m}_1, \ldots, \mathbf{m}_L)]$. The linearity of the Θ and E_{cv} ensures that each codeword is the bitwise sum of L codewords, i.e.,

$$\rho(\mathbf{m}_1, \ldots, \mathbf{m}_L) = E_{\text{cv}}\left(\sum_{j=1}^{L} \widetilde{\mathbf{m}}_j\right) = \sum_{j=1}^{L} E_{\text{cv}}(\widetilde{\mathbf{m}}_j).$$

Let $\widetilde{\mathbf{a}}_l$ denote the realization of $\widetilde{\mathbf{m}}_l$ for $l \in \mathsf{S}_j$. The subcode $\mathscr{C}_{\mathbf{a}_l}$ is a coset of the subcode generated by the other $L-1$ messages, i.e.,

$$\begin{aligned}\mathscr{C}_{\mathbf{a}_l} &= \rho(\mathbf{m}_1, \ldots, \mathbf{a}_i, \ldots, \mathbf{m}_L) \\ &= E_{\text{cv}}(\widetilde{\mathbf{a}}_l) + \left\{ \sum_{i=1, i \neq l}^{L} E_{\text{cv}}(\widetilde{\mathbf{m}}_i) \ : \ \mathbf{m}_i \in \{0,1\}^K \right\}. \qquad (1)\end{aligned}$$

At the jth receiver, using the side information, $\sum_{l \in \mathsf{S}_j} E_{\text{cv}}(\widetilde{\mathbf{a}}_l)$ is computed and subtracted from the received codeword. Then, it needs to perform ML decoding in the subcode

$$\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}\Big|_{\mathbf{a}_{\mathsf{S}_j}=0} = \left\{ \sum_{i=1, i \notin \mathsf{S}_j}^{L} E_{cv}(\widetilde{\mathbf{m}}_i) \ : \ \mathbf{m}_i \in \{0,1\}^K \right\}.$$

All subcodes $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}$ are cosets of $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}\big|_{\mathbf{a}_{\mathsf{S}_j}=0}$ and the Hamming distance distribution of each subcode is same as that of $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}\big|_{\mathbf{a}_{\mathsf{S}_j}=0}$. Therefore, the minimum Hamming distance of each subcode is independent of the actual bit value of the side information.

B. Structure of the Channel Encoder

We now give the structure of the convolutional encoder E_{cv} suitable for TCMIR. Some properties of convolutional encoders are discussed when side information is available during decoding. These properties indicate how side information affects the decoder to estimate the transmitted codewords and help to identify the convolutional encoders that can maximally utilize the side information.

Let ν be the number of shift registers in the convolutional encoder. Each trellis block will have 2^ν states and 2^k branches will stem out of each state. At a particular receiver and at the trellis blocks with side information[1], only 1 transition stems from each state since the exact value of the input (side information) is known. The following proposition gives the structure of the set of output sequences corresponding to a given realization of side information.

Proposition 1. *The set of 2^ν binary output sequences corresponding to the all-zero input to a convolutional encoder forms a commutative group under bitwise addition and the sets of 2^ν binary output sequence corresponding to other inputs are cosets of this group.*

Proof. Each branch is labeled with an n bit output $\mathbf{y} = (y_1 y_2 \dots y_n)$. Let b_j and T_l denote the jth input and the lth shift register bit respectively. Then, $\mathbf{y}$ can be written as a linear combination of $k+\nu$ bits as follows:

$$\mathbf{y} = \mathbf{b}\mathbf{G}_1 + \mathbf{t}\mathbf{G}_2 = (b_1, b_2, \dots, b_k)\,\mathbf{G}_1 + (T_1, T_2, \dots, T_\nu)\,\mathbf{G}_2,$$

where $\mathbf{G}_1 \in \mathbb{F}_2^{k\times n}$ and $\mathbf{G}_2 \in \mathbb{F}_2^{\nu\times n}$. The set of 2^ν possible state vectors $\mathbf{t}$ form a commutative group under bitwise addition. Then, for $\mathbf{b} = 0$, the set $\mathcal{G} = \{\mathbf{y} = \mathbf{t}\mathbf{G}_2 : \mathbf{t} \in \mathbb{F}_2^\nu\}$ forms a commutative group under bitwise addition. Moreover, for each $\mathbf{b} \neq 0$, the set $\mathcal{H}_\mathbf{b} = \mathbf{b}\mathbf{G}_1 + \mathcal{G}$. □

This theorem indicates that the subcode of $\mathscr{C}$ corresponding to $\mathbf{m}_l = 0$ for all $l \in \mathsf{S}_j$ is a binary linear code and the subcodes corresponding to $\mathbf{m}_l \neq 0$ for at least one $l \in \mathsf{S}_j$ are cosets of this linear subcode.

Remark 1. *If $\mathcal{G}$ is a proper subgroup of $\mathbb{F}_2^n$, then $\mathcal{G}$ and its cosets form a nontrivial partition of $\mathbb{F}_2^n$. In this case, the estimation of the most likely transmitted signal (or equivalently, output sequence) corresponding to the trellis block with side information will require comparison between the received signal with strictly fewer than 2^n signal points (i.e., points corresponding only to the elements of the coset corresponding to the side information realization). This affords distance gain in signal mapping. If $\mathcal{G} = \mathbb{F}_2^n$, the partition is trivial and the estimation of the most likely transmitted signal for the trellis block with side information will require comparison between the received signal with each of the 2^n signal points. In this case, no distance gain in signal mapping is achieved even in the presence of side information. An example of a convolutional encoder with such trivial partition is a rate $^1/_2$ binary convolutional encoder with $\nu = 4$ and the generator sequence $[23\ 35]_8$ [13]. The set $\mathcal{G}$ of 2^4 output sequences corresponding to all-zero input are $\{00, 01, 10, 11\} = \mathbb{F}_2^2$.*

The following corollary shows that convolutional encoders with systematic bits provide a nontrivial partition.

[1]In the context of decoding at a particular receiver, by "trellis block with side information" we mean the trellis block when the input to the convolutional encoder in known at the receiver.

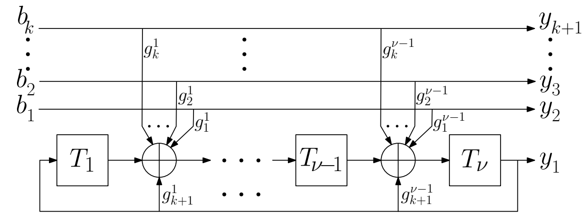

Fig. 3. A systematic $^k/_{(k+1)}$ convolutional encoder with feedback.[2]

Corollary 1. *For a convolutional encoder with at least one systematic bit, the set of 2^ν binary output sequences corresponding to the all-zero input is a proper subgroup of $\mathbb{F}_2^n$.*

Proof. Without loss of generality, let the last ℓ bits $(1 \leqslant \ell \leqslant k)$ of the output be systematic. Then, the last ℓ bits of each $\mathbf{y} \in \mathcal{G}$ will be 0. Hence $\mathcal{G}$ will be a proper subset of $\mathbb{F}_2^n$. Since $\mathcal{G}$ is a commutative subgroup (by Proposition 1), its is a proper subgroup of $\mathbb{F}_2^n$. In this case, $|\mathcal{G}| \leqslant 2^{n-\ell}$ and the decoding will require at most $2^{n-\ell}$ comparisons (by Remark 1). If $\ell = k$ and $n = k+1$, each coset will have only two elements. □

Since the proposed TCM scheme is designed for AWGN channel, the bit labeling must satisfy *Ungerboeck's set partitioning rules* [8]. The structure of a convolutional encoder with maximum number of systematic bits ($\ell = k = n-1$) and that satisfies Ungerboeck's rules is given in Fig. 3.

Corollary 2. *For the convolutional encoder of Fig. 3, the set of output sequences corresponding to any particular input has only two distinct elements.*

Proof. For the input $(b_1, b_2, \dots, b_k)$, the possible outputs (depending upon the current state of the shift registers) are $(0, b_1, b_2, \dots, b_k)$ and $(1, b_1, b_2, \dots, b_k)$. □

Decoding errors (*error events*) are caused by the path chosen by the decoder does not coincide with the correct path (path corresponding to the transmitted codeword), i.e., the chosen path occasionally diverges from the correct path at some time instants and remerges at a later time [9]. The minimum among the Euclidean distances between all such pairs of paths is denoted by d_{free}. In Theorem 1, we show that an error event cannot end at the trellis blocks with side information for the convolutional encoder of Fig. 3.

Theorem 1. *For the trellis blocks corresponding to the side information and every realization of the side information, the transition of current states to next states is bijective for the convolutional encoder in Fig. 3.*

Proof. The transition from current states to next states is surjective. We now show that the transition is also injective.

[2]Coefficient $g_i^j = 1$ where $i \in 1, 2, \cdots, k$ and $j \in 1, 2, \cdots, \nu-1$ denotes that there is a connection between the ith input bit and the jth adder and $g_i^j = 0$ otherwise. Likewise, coefficient $g_{k+1}^j = 1$ denotes that there is a connection between the feedback and the jth adder. The structure of a specific convolutional encoder is represented by $[\mathbf{g}_k\, \mathbf{g}_{k-1} \cdots \mathbf{g}_1]$. Each term is a binary sequence: $\mathbf{g}_m = \{(g_m^1 \cdots g_m^{\nu-1}) | m \in \{1, 2, \cdots, k+1\}\}$. In this paper, $\mathbf{g}_m$ are displayed in octal format and the rightmost bit is the least significant bit.

Let $T_1, T_2, \ldots, T_\nu$ represent the bits stored in shift registers and $b_1, b_2, \cdots, b_k$ denote k input bits of current trellis block. A trellis state is represented by the vector $(T_1, T_2, \ldots, T_\nu)$. Let $\psi(T_i)$ be the value of ith shift register in the next state. Then,

$$\psi(T_i) = \begin{cases} T_{i-1} + g_{k+1}^{i-1} T_\nu + \left(\sum_{j=1}^{k} g_j^{i-1} b_j\right) & i \in \{2, \ldots, \nu\} \\ T_\nu & i = 1. \end{cases}$$

Let $(b_1, b_2, \ldots, b_k)$ be the input sequence and $(T_1, T_2, \ldots, T_\nu)$ and $(T_1', T_2', \ldots, T_\nu')$ be any two distinct current states. Let $(\psi(T_1), \ldots, \psi(T_\nu)) = (\psi(T_1'), \ldots, \psi(T_\nu'))$. Then, by the above definition of $\psi(T_i)$, $T_\nu = T_\nu'$ and $T_{i-1} + g_{k+1}^{i-1} T_\nu + \sum_{j=1}^{k} g_j^{i-1} b_j = T_{i-1}' + g_{k+1}^{i-1} T_\nu' + \sum_{j=1}^{k} g_j^{i-1} b_j$ or $T_{i-1} = T_{i-1}'$ for all $i \in \{2, \ldots, \nu\}$. Thus, ψ is injective also. □

This theorem shows that any two paths cannot merge at the trellis blocks with side information. In Viterbi decoding, the decoder does not have to compare the accumulated path metric of each state at these blocks. Therefore, error events which could have ended at these blocks when no side information is available can be avoided.

C. Bit Labeling of TCMIR

We now propose the bit labeling scheme for TCMIR. Corollary 2 shows that 2^{k+1} output sequences are partitioned into 2^k subsets, denoted by $P_1, P_2, \ldots, P_{2^k}$. For the trellis block with side information, there are only two candidates to choose from for the transmitted constellation point. Let d_i be the Euclidean distance between the two constellation points mapped to by the two elements of P_i, $1 \leqslant i \leqslant 2^k$. The bit labeling must satisfy the following rules:

(R1) The bit labeling should follow the set partitioning rules.
(R2) All d_is are identical, i.e, $d_1 = d_2 = \cdots = d_{2^k}$.
(R3) d_i is maximized for each i, $1 \leqslant i \leqslant 2^k$.

Signal mappings satisfying (R1) yield TCM schemes that have good error performance in AWGN channel, (R2) guarantees that the distance gain in signal mapping is independent of the actual bit value of the side information, and (R3) maximizes the distance gain that can be achieved by side information. A bit labeling of the normalized 8PSK that does not satisfy (R1)-(R3) is illustrated in Fig. 4(a) where each point is labeled with $y_3y_2y_1$. When the decoder knows $y_3y_2 = 11$, the minimum Euclidean distance is increased from $d_0 = 0.765$ to $d_S^{11} = 1.848$. However, no distance gain is achieved if $y_3y_2 = 01$.

The proposed 8PSK labeling for rate $^2/_3$ TCMIR is depicted in Fig. 4(b). When the value of y_3y_2 is known to the decoder, the minimum Euclidean distance between constellation points is always increased from d_0 to d_S. Since this bit labeling follows set partitioning rules (solid and hollow circles are distinguished by the parity bit) [8], the error performance of a receiver without side information will be same as that of conventional TCM scheme using this bit labeling.

IV. Performance Analysis and Code Search

The performance analysis of TCMIR based on d_{free} is presented in this section. *Scalar transfer function* [9] is used in

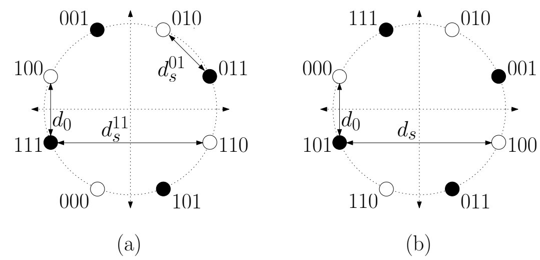

Fig. 4. 8PSK bit labeling that (a) does not satisfy (R1)-(R3) and (b) that satisfies (R1)-(R3).

the performance analysis to evaluate d_{free} of TCMIR scheme for receivers with and without side information. The code search of good TCMIR schemes is conducted by choosing the convolutional encoders whose TCMIR schemes have the largest d_{free}.

A. Performance Analysis Based on the Transfer Function

For a TCM scheme, d_{free} can be obtained by *error state diagram* [9]. The error state diagram of a TCM scheme is determined by its convolutional encoder. The labels of TCMIR's error state diagram for receivers without side information are discussed below. Due to the nonlinear mapping, labels of branches of the error state diagram are $2^\nu \times 2^\nu$ matrices [10]; the label of a branch corresponding to the binary error $\mathbf{e}_i$ is $G(\mathbf{e}_i)$. If there is no transition from state p to q in the code trellis, the entry $G(\mathbf{e}_i)_{p,q}$ is zero. Other entries are given by

$$G(\mathbf{e}_i)_{p,q} = \frac{1}{2^k} \mathbf{D}^{\|f(\mathbf{c}_{p\to q}) - f(\mathbf{c}_{p\to q} \oplus \mathbf{e}_i)\|^2} \quad (2)$$

where $\mathbf{D}$ is a dummy variable, $\mathbf{c}_{p\to q}$ is the output sequence generated by the transition from p to q, and $f(\cdot)$ is the constellation mapping.

If the sum of all elements of each row of each $G(\mathbf{e}_i)$ is same, scalar labels can be substituted for the corresponding matrix labels and reduce the computation complexity of computing the transfer function of TCMIR. The matrix $G(\mathbf{e}_i)$ is then represented by the scalar $\omega(\mathbf{e}_i)$, which is the summation of the elements of any row of $G(\mathbf{e}_i)$, i.e.,

$$\omega(\mathbf{e}_i) = \sum_q G(\mathbf{e}_i)_{p,q}. \quad (3)$$

Let $\mathbf{C}_0$ be the set of 2^k binary output sequences associated with the all-zero state of the convolutional encoder. A sufficient condition [10] for the above substitution is that there exists a one-to-one mapping $\Phi : f(\mathbf{s}) \mapsto (\mathbf{s} \oplus \tilde{\mathbf{s}})$ where $\mathbf{s} \in \mathbf{C}_0$ and $\tilde{\mathbf{s}}$ is a constant binary sequence. For the bit labeling of Fig. 4(b), one such one-to-one mapping is given by $\Phi(f(\mathbf{c})) = f(\mathbf{c} \oplus 101)$, which is the clockwise rotation by $\pi/4$. Therefore, labels $G(\mathbf{e}_i)$ can be replaced by scalar labels $\omega(\mathbf{e}_i)$ in further analysis.

For the error state diagram of a receiver with side information, L consecutive trellis blocks are concatenated to form a new single trellis block. The label $G(\mathbf{E}_i)$ of the transition from state p to q, where $\mathbf{E}_i = [\mathbf{e}_{i+1}\mathbf{e}_{i+2} \ldots \mathbf{e}_{i+L}]$ (concatenation of errors in L consecutive trellis blocks), is

$$G(\mathbf{E}_i)_{p,q} = 2^{-Lk} \sum \mathbf{D}^{\|f(\mathbf{c}_{p\to q}) - f(\mathbf{c}_{p\to q} \oplus \mathbf{E}_i)\|^2} \quad (4)$$

where summation accounts for possible parallel transitions in the new trellis block. For the jth receiver with side information S_j, the subcode $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}$ is a coset of $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}\big|_{\mathbf{a}_{\mathsf{S}_j}=0}$ (by (1)). Therefore, the set of binary error events of the subcode $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}$ is $\mathscr{C}_{\mathbf{a}_{\mathsf{S}_j}}\big|_{\mathbf{a}_{\mathsf{S}_j}=0}$, i.e., the error state diagram when side information is $\mathbf{a}_l\ :\ l \in \mathsf{S}_j\}$ is identical to the trellis diagram with input $\{\mathbf{m}_l$ set to zero for all $l \in \mathsf{S}_j\}$.

Let $\mathbf{x}=(x_0,\ldots,x_{2^\nu-1})^\mathsf{T}$ and $\mathbf{x}_0=(0,L_{0,1},\ldots,L_{0,2^\nu-1})^\mathsf{T}$, where $x_0,\cdots,x_{2^\nu-1}$ are transfer functions of paths ending at trellis states $0,\ldots,2^{\nu-1}$ respectively and $L_{i,j}$ is the matrix (scalar) label of transition from state i to state j defined in (2) (respectively (3)). Then, the transfer function describing all paths diverging from state 0 and remerging to state 0 can be derived using *state transition matrix* $\mathbf{T}$ by solving the following equation [9]:

$$\mathbf{x} = \mathbf{T}\mathbf{x} + \mathbf{x}_0 \tag{5}$$

Entries of $\mathbf{T}$ are (2) when there is no side information and (4) when there is side information; (2) and (4) can be simplified to scalar labels using (3). The solution of (5)

$$\mathbf{x} = (\mathbf{I} - \mathbf{T})^{-1}\mathbf{x}_0 = (\mathbf{I} + \mathbf{T} + \mathbf{T}^2 + \cdots)\mathbf{x}_0 \tag{6}$$

gives the final transfer function of each state. The transfer function which enumerates all Euclidean distances between the all-zero codeword and its error events is given by x_0 in the final solution of $\mathbf{x}$. The transfer function is

$$\mathbf{T}(D) = \sum_i N(d_i)D^{d_i^2} \tag{7}$$

where $N(d_i)$ is the average number of error events that have Euclidean distance d_i in the trellis diagram.

B. Code Search of Good TCMIR Schemes

We performed code search for good TCMIR schemes when $L = 2$ messages are to be broadcast and messages are encoded by a rate $^2/_3$ convolutional encoder of the form given in Fig. 3. The parameter considered in the code search for possible TCMIR scheme is d_{free}, which is computed based on the transfer function analysis discussed in Section IV-A.

Two numerical approximation are applied to efficiently extract d_{free}. First, the closed form expression of (6) requires the computation of matrix inverse of $(\mathbf{I} - \mathbf{T})$. Since d_{free} is the main parameter that affects the performance, only first few terms of (6) were used in further computations. According to [8], the number of terms is chosen to be twice of the number of shift registers ν in our computations. Second, to extract the minimum exponent d^2_{free} in the summation (7), an approximation $\phi(D)$ was proposed in [12],

$$\phi(D) = \frac{\mathbf{T}(eD)}{\mathbf{T}(D)}, \tag{8}$$

where e is the base of natural logarithm. This function decreases to the limit d^2_{free} as $D \to 0$.

After applying these numerical approximation, we did code search for TCMIR schemes with number of shift registers from $\nu = 3$ to $\nu = 6$. Complete list of good codes is given in the appendix. The minimum squared Euclidean distances of these TCMIR schemes when a receiver knows and does not know side information are also provided. It can be observed that, codes which have the best performance for receivers with side information may not have the best performance for the receivers without any side information.

V. Simulation Results

We select one TCMIR scheme among best side information codes for $L = 2$ and $\nu = 4$ and compare its BER with the nested code. In this rate $^2/_3$ TCMIR scheme, two information bits of each message are alternately fed into the 16-state convolutional encoder $[\mathbf{g}_3\mathbf{g}_2\mathbf{g}_1] = [3\,2\,7]$ and every three output bits are mapped to an 8PSK signal point using the labeling of Fig. 4(b). In the nested code, information bits of each message are encoded by their corresponding generator matrices $\mathbf{G}_L$ and $\mathbf{G}_R$ [5]. The output codeword is the bitwise XOR of codewords $\mathbf{m}_1\mathbf{G}_L$ and $\mathbf{m}_2\mathbf{G}_R$ which is then mapped to a point in 8PSK with Gray mapping. Since the code rate of the convolutional code in the selected TCMIR scheme is $^2/_3$, we choose the rate $^2/_3$ 256-state convolutional code from [5] for comparison. For the receiver with no side information, decoding is based on the rate $^2/_3$ convolutional code. When information bits of one of the two messages is known to a receiver, decoding is equivalent to decoding a rate $^1/_3$ convolutional code with generator matrix $\mathbf{G}_L$ and this low rate nested code has the same number of shift registers as the selected TCMIR scheme. For both the schemes, 1.2×10^7 coded bits are generated and at least 100 bit errors are accumulated for BER analysis. We use soft input Viterbi decoder to decode the received codewords. The information spectral efficiency is 1 b/dim when for a receiver with no side information and 0.5 b/dim when one message is known as the side information. The BER of the two schemes are shown in Fig. 5. For a receiver without side information, the TCMIR scheme is 0.3 dB worse than the nested code at BER of 10^{-4}. When one messages is known as the side information, our scheme outperforms the nested code by around 2.1dB at BER of 10^{-4}. Upper and lower bounds on BER for the TCMIR scheme are also provided in Fig. 5 to verify the code search in Section IV-A. For the TCMIR scheme, the bound analysis for a receiver without side information follows the same approach as that of the conventional TCM [9, pp. 111,116]. The bound analysis for TCMIR with side information uses a similar approach but the entries of the state transition matrix $\mathbf{T}$ are substituted by those defined in (4).

VI. Discussion and Conclusion

We have presented a coding scheme for informed receivers. This coding scheme involves joint design of channel encoder and modulator. For a receiver with side information, the performance improvement of the receiver is independent of actual bit value of the side information. Simulation shows that TCMIR has larger side information gain when compared with nested codes.

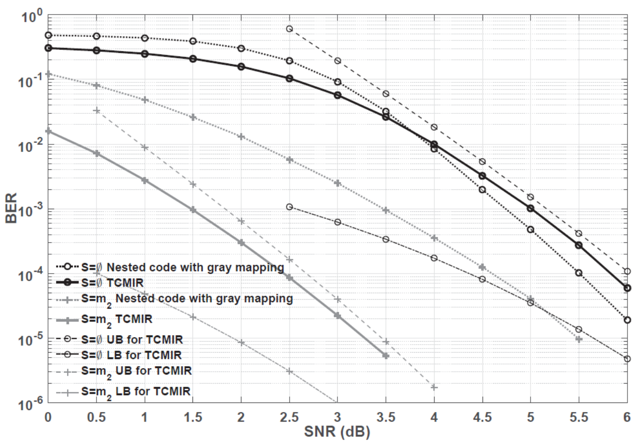

Fig. 5. BER of the nested code with 8PSK Gray mapping and equivalent TCMIR scheme.

In this paper, we provide good TCMIR schemes when 2 messages are to be communicated. In general, if L messages are to be communicated, there are $2^L - 1$ possible side information sets (we exclude the case in which all the messages are known a priori). To find the TCMIR schemes which perform well for all the cases, the error performance of each case should be analyzed based on the approach followed in Section IV-A. This may result in exponential growth of complexity of code search. However, this limitation also applies to previous code design [4]–[7].

Possible extensions of the current TCMIR scheme include the efficient code construction for more messages case, bit labeling schemes for higher order and multidimensional modulation, and TCMIR schemes designed for fading channel.

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APPENDIX
GOOD CODES FOR $L = 2$ AND $k = 2$

The octal representation of convolutional encoders of good TCMIR schemes for different shift register length are listed below. Following the notation of Fig. 2, the structure of each convolutional encoder is specified by $[\mathbf{g}_3\ \mathbf{g}_2\ \mathbf{g}_1]$. We call the TCMIR schemes with the largest squared Euclidean distance without side information as *best original codes* and the schemes with the largest distance gain as *best side information codes*. The notation $a \rightarrow b$ represents that the squared Euclidean distance d^2_{free} of the TCMIR scheme is a for receivers without side information and b for receivers with side information.

1) $\nu = 3$

Best codes
- $\mathbf{G}$: $[0, 1, 2], [0, 1, 3], [0, 2, 1], [0, 2, 3]$
- Squared distance gain: $4.58 \rightarrow 7.17$ or $1.95\,\text{dB}$

2) $\nu = 4$

Best original codes:
- $\mathbf{G}$: $[5, 1, 6], [5, 1, 7], [5, 4, 3], [5, 4, 7], [7, 1, 6], [7, 1, 7], [7, 4, 3], [7, 4, 7]$
- Squared distance gain: $5.17 \rightarrow 7.41$ or $1.56\,\text{dB}$

Best side information codes:
- $\mathbf{G}$: $[3, 2, 5], [3, 2, 7], [6, 2, 5], [6, 2, 7]$
- Squared distance gain: $4.58 \rightarrow 10$ or $3.38\,\text{dB}$

3) $\nu = 5$

Best original codes:
- $\mathbf{G}$: $[12, 7, 11], [12, 7, 16], [5, 16, 7], [5, 16, 11]$
- Squared distance gain: $5.76 \rightarrow 10$ or $2.40\,\text{dB}$

Best side information codes:
- $\mathbf{G}$: $[0, 2, 11], [0, 2, 13], [0, 4, 11], [0, 4, 15], [2, 2, 15], [2, 2, 17], [4, 4, 13], [4, 4, 17]$
- Squared distance gain: $4.58 \rightarrow 12$ or $4.18\,\text{dB}$

4) $\nu = 6$

Best original codes:
- $\mathbf{G}$: $[15, 36, 5], [15, 36, 33], [26, 17, 24], [26, 17, 33]$
- Squared distance gain: $6.34 \rightarrow 12$ or $2.77\,\text{dB}$

Best side information codes:
- $\mathbf{G}$: $[1, 4, 23], [1, 4, 27], [6, 2, 25], [6, 2, 27], [14, 10, 25], [14, 10, 35], [20, 4, 31], [20, 4, 35]$
- Squared distance gain: $4.58 \rightarrow 13.17$ or $4.58\,\text{dB}$