# Correspondence

## Performance of High-Diversity Multidimensional Constellations

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***Abstract*—Following the approach introduced by Cavers and Ho, the performance of component-interleaved multidimensional constellations over the Rayleigh fading channel is evaluated analytically. Error probabilities are approximated by the union bound using an exact expression of the pairwise error probability. Simulation results show that this bound can be used effectively as a design criterion for the selection of high-diversity multidimensional constellation over the Rayleigh fading channel.**

***Index Terms*—Component-interleaving, modulation diversity, Rayleigh fading channel, union bound.**

### I. Introduction

The concept of *modulation diversity* was introduced in [2] as a way to enhance the error performance of communication systems over the Rayleigh fading channel. Several multidimensional signal sets with high diversity order have been proposed to obtain substantial gains over conventional modulation schemes in a fading environment [3]–[5]. Modulation diversity can also be viewed as a special form of signal space coding producing highly efficient (in terms of bandwidth and power) schemes over the fading channel [6].

The modulation diversity order of a signal set is defined as the minimum number of *different* components between any two distinct points of the set. This definition applies to every modulation scheme and affects its performance over the fading channel in conjunction with component interleaving. By use of component interleaving, fading attenuations over different space dimensions become statistically independent. An attractive feature of these schemes is that we have an improvement of error performance without even requiring the use of conventional channel coding.

As for other types of diversity such as space, time, frequency, and code diversity, for increasingly high modulation diversity order, the performance over the fading channel approaches that achievable over the Gaussian channel [7]. Thus we may say that the fading channel is converted into a Gaussian channel (to a certain extent) and coding schemes which are good for the Gaussian channel can be applied to constellations with high diversity order.

One possible way to obtain multidimensional constellations of high diversity order is to resort to constructions based on the *canonical embedding* of algebraic number fields, which were first considered in [2]–[5]. Another approach is based on the rotation of conventional multidimensional constellations (see [6]).

Most of the research work done in the area of multidimensional rotated constellation uses the Chernoff bound for performance analysis [2]–[8]. This provides simple design criteria for good signal sets. However, the looseness of the Chernoff bound prevents us from obtaining precise performance results so that, in most cases, simulation is needed. A first step in improving the accuracy of error performance analysis has been made by the authors in [9] focusing on two-dimensional schemes. In this work we extend the analysis to multidimensional constellations and provide additional insight and some optimality criteria that can be helpful in the design for the Rayleigh fading channel.

The approach followed here is directly derived from [1]. The main difference here is that we obtain a closed-form expression of the pairwise error probability in terms of the parameters of the pairwise error event.

The system model considered here is described in Section II. Then, we briefly review the concepts of pairwise error probability (PEP) and Chernoff bound with reference to the application considered here. In Section III we calculate the exact expression of the PEP for any multidimensional constellation with coordinate-independent Rayleigh fading. Finally, we apply this exact expression of the PEP to calculate the union bound on the error probability of some rotated multidimensional constellations considered in [6]. Results obtained are compared to those obtained by simulation and by use of the Chernoff bound in the union bound expression.

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### II. System Model

We consider a communication system with ideal (infinite-depth) component interleaving over a Rayleigh fading channel, unaffected by intersymbol interference or any impairment other than additive white Gaussian noise (AWGN). Let $\boldsymbol{x} = (x_1, x_2, \cdots, x_n)$ denote a transmitted signal vector from a given $n$-dimensional constellation $S$. Received signal samples are then given by

$$y_i = g_i x_i + n_i, \qquad \text{for } i = 1, 2, \cdots, n.$$

The coefficients $g_i$ are (real) independent Rayleigh-distributed random variables with unit second moment (i.e., $E[g_i^2] = 1$) representing the fading coefficients and $n_i$ are real Gaussian random variables with mean zero and variance $N_0/2$ representing the additive noise. We also set

$$\boldsymbol{g} = (g_1, g_2, \cdots, g_n)$$
$$\boldsymbol{n} = (n_1, n_2, \cdots, n_n)$$

and

$$\boldsymbol{y} = (y_1, y_2, \cdots, y_n)$$

and write, more compactly, $\boldsymbol{y} = \boldsymbol{g} \odot \boldsymbol{x} + \boldsymbol{n}$.

The coefficients $g_i$'s are independent because of the infinite component interleaver used. Perfect phase tracking and channel state information (CSI) are assumed at the receiver, which performs maximum-likelihood (ML) detection. The receiver computes the sample metrics

$$m(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{g}) = ||\boldsymbol{y} - \boldsymbol{g} \odot \boldsymbol{x}||^2 \quad \forall \boldsymbol{x} \in S \tag{1}$$

where $||\cdot||^2$ is the standard squared Euclidean distance, and chooses the signal $\hat{\boldsymbol{x}}$ attaining the minimum value of $m(\boldsymbol{x}|\boldsymbol{y}, \boldsymbol{g})$. Error performance can be evaluated by use of the Chernoff bound as is done in [2]–[8] for multidimensional signal sets. These works show that the PEP is determined asymptotically by the *modulation diversity order*, $L$, and the *minimum $L$-product distance*, $d_{p,\min}^{(L)}$, of the constellation.

The former is given by the minimum number of different components in all possible pairs of points in the signal set. The latter is defined as

$$d_{p,\min}^{(L)} = \min_{\boldsymbol{x},\hat{\boldsymbol{x}}} \prod_{x_i \neq \hat{x}_i} |x_i - \hat{x}_i|.$$

Good signal sets have high $L$ and $d_{p,\min}^{(L)}$.

## III. Performance Analysis

A standard approach to error performance analysis consists of evaluating the symbol error probability of a signal set $S$ by using the union bound

$$P(e) \leq \frac{1}{|S|} \sum_{\boldsymbol{x}\in S} \sum_{\boldsymbol{x}\neq\hat{\boldsymbol{x}}} P(\boldsymbol{x} \to \hat{\boldsymbol{x}}). \tag{2}$$

Each PEP $P(\boldsymbol{x} \to \hat{\boldsymbol{x}})$ is commonly approximated by using the Chernoff bound or other upper bound. However, as we show in the following theorem, the PEP can be calculated exactly over the Rayleigh fading channel with component interleaving.

The approach followed here is derived from [1]. We obtain a less general but closed-form expression of the pairwise error probability in terms of the parameters of the pairwise error event.

*Theorem:* Let us define

$$\begin{aligned} \boldsymbol{x} &= (x_1, x_2, \cdots, x_n) \\ \hat{\boldsymbol{x}} &= (\hat{x}_1, \hat{x}_2, \cdots, \hat{x}_n) \\ \delta_i &= |x_i - \hat{x}_i| \neq 0, \qquad \text{for } i = 1, 2, \cdots, n \end{aligned}$$

and let $\boldsymbol{x} \to \hat{\boldsymbol{x}}$ represent a pairwise error event. Additionally, assume that all $\delta_i$'s are distinct. The pairwise error probability is then given by

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = \frac{1}{2}\sum_{i=1}^{n}\left(1 - \frac{\delta_i}{\sqrt{4N_0 + \delta_i^2}}\right) \prod_{\substack{j=1\\ j\neq i}}^{n} \frac{\delta_i^2}{\delta_i^2 - \delta_j^2}. \tag{3}$$

*Proof:* Using the receiver metric given in (1), we can write the PEP as

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = P\left(\sum_{i=1}^{n} |y_i - g_i\hat{x}_i|^2 < \sum_{i=1}^{n} |y_i - g_i x_i|^2\right) = P(\Delta < 0)$$

with $\Delta = \sum_{i=1}^{n} \Delta_i$ and

$$\Delta_i = |g_i|^2 |x_i - \hat{x}_i|^2 - 2\, g_i (x_i - \hat{x}_i) n_i.$$

Streamlining the approach followed in [1], we obtain the Laplace inversion formula

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \Phi_\Delta(s)\, \frac{ds}{s} \tag{4}$$

where $c > 0$ is in the region of convergence of the characteristic function $\Phi_\Delta(s)$ defined as

$$\Phi_\Delta(s) = E[e^{-s\Delta}] = \prod_{i=1}^{n} \Phi_{\Delta_i}(s).$$

After straightforward algebra, we obtain

$$\Phi_{\Delta_i}(s) = E[e^{-s\Delta_i}] = \frac{1}{1 + s(1 - sN_0)\delta_i^2}.$$

Then, we evaluate the integral (4) as a sum of residues

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = -\sum_{\sigma_i:\, \mathrm{Re}\,[\sigma_i]>0} \text{Residue}\, [\Phi_\Delta(s)/s;\ s = \sigma_i] \tag{5}$$

where the $\sigma_i$'s are simple poles of $\Phi_\Delta(s)$ since we assumed that the $\delta_i$ are distinct. A further step (not made in [1]) consisting in the evaluation of the residues provides the unexpectedly simple result of this theorem. In fact, the poles to be considered are $\sigma_0 = 0$ and

$$\sigma_i = \frac{1}{2N_0}\left[1 \pm \sqrt{1 + 4N_0/\delta_i^2}\right], \qquad \text{for } i = 1, \cdots, n.$$

The residues in (5) are given by

$$\begin{aligned} &\text{Residue}\left[\frac{\Phi_\Delta(s)}{s};\ s = \sigma_i\right]_{\mathrm{Re}\,[\sigma_i]>0} \\ &\quad = \frac{2N_0\delta_i}{\delta_i + \sqrt{\delta_i^2 + 4N_0}} \cdot \frac{-1}{\delta_i\sqrt{\delta_i^2 + 4N_0}} \cdot \prod_{j\neq i} \frac{\delta_i^2}{\delta_i^2 - \delta_j^2} \end{aligned} \tag{6}$$

for positive $\sigma_i$. The asserted result (3) is obtained by inserting (6) into (5). □

The validity of the above theorem is apparently limited to the case of full diversity signal sets $(L = n)$ and all distinct $\delta_i$. However, the other cases can be dealt with after considering the following remarks.

*Remark 1:* If $L < n$, with the previous notations, let $\delta_i$ for $i = 1, \cdots, L$ be the nonzero elements, then we have

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = \frac{1}{2}\sum_{i=1}^{L}\left(1 - \frac{\delta_i}{\sqrt{4N_0 + \delta_i^2}}\right) \prod_{\substack{j=1\\ j\neq i}}^{L} \frac{\delta_i^2}{\delta_i^2 - \delta_j^2}. \tag{7}$$

As already observed in [1], it is interesting to note how this expression is totally independent of the signal space dimension $n$. Actually, it only depends on the modulation diversity $L$ between $\boldsymbol{x}$ and $\hat{\boldsymbol{x}}$ and the nonzero component distances $\delta_i$.

*Remark 2:* We verified numerically that the apparent discontinuities in (3) arising when some $\delta_i$ are equal can be removed by small perturbations of the equal terms without affecting significantly the numerical result obtained. Although, in principle, we can extend the analytical approach of the previous theorem to this case, it turns out to be rather difficult to compute the residues of poles with multiplicity greater than one, and the resulting expression of the PEP is too complex for practical applications.

*Remark 3:* It can easily be shown, from a symmetric argument, that the minimum PEP under the constraint of constant $\sum_{i=1}^{L} \delta_i^2$ is obtained when all the $\delta_i$'s are equal. In this case, the PEP can be evaluated by calculating a single residue of a multiple pole (with multiplicity $L$). Setting $\delta_i = \delta$ for $i = 1, \cdots, L$, after straightforward algebra, we obtain

$$\begin{aligned} P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = \frac{1}{2} - \frac{1}{2}(1 + 4N_0/\delta^2)^{-L+1/2} \\ \cdot \sum_{i=0}^{L-1} \binom{L - 1/2}{i} (4N_0/\delta^2)^i. \end{aligned} \tag{8}$$

Fig. 1 shows this optimal PEP for several values of diversity $L$ under the constraint of constant distance $L\delta^2$ with

$$\hat{\boldsymbol{x}} = -\boldsymbol{x} = (\delta/2, \cdots, \delta/2).$$

The figure shows that, as diversity increases, the error performance approaches that of binary PAM over the AWGN channel.

*Remark 4:* We can write the asymptotic expansion (as $N_0 \to 0$) of (7) as

$$P(\boldsymbol{x} \to \hat{\boldsymbol{x}}) = \frac{\gamma_L}{[d_p^{(L)}]^2} N_0^L + O(N_0^{L+1}), \qquad N_0 \to 0 \tag{9}$$

where $d_p^{(L)} = \prod_{i=1}^{L} \delta_i$ and

$$\gamma_L = 2^{L-1}(2L-1)!!/L! = 3, 10, 35, 126, 462, 1716, \cdots, \quad \text{for } L = 1, 2, 3, 4, 5, 6, \cdots$$

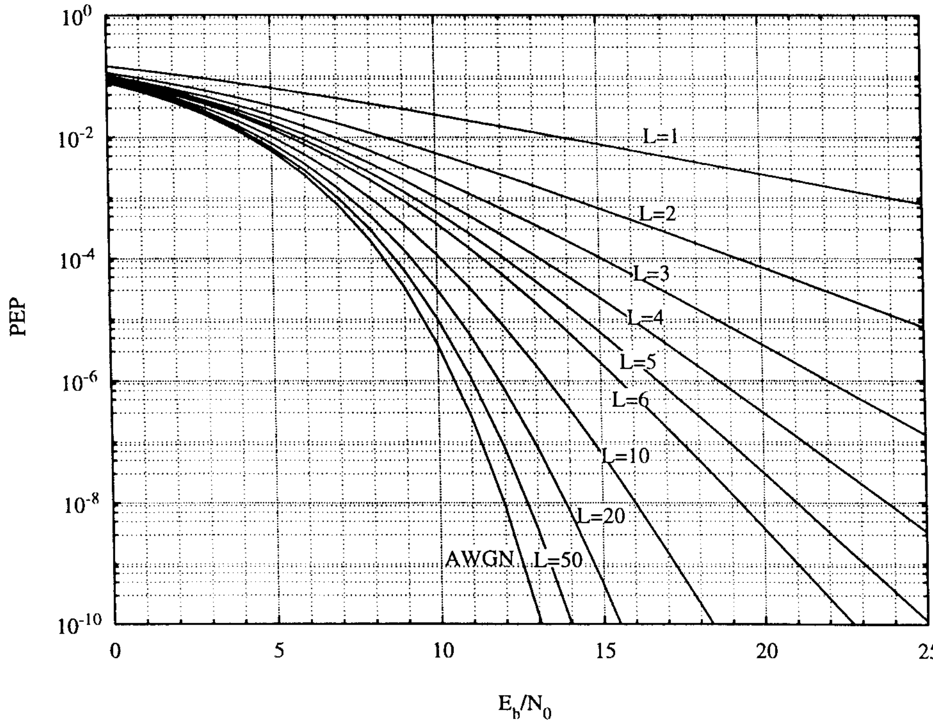


Fig. 1. Comparison of pairwise error probabilities, evaluated by (8), for several values of diversity $L$ over the Rayleigh fading channel and binary PAM over the AWGN channel.

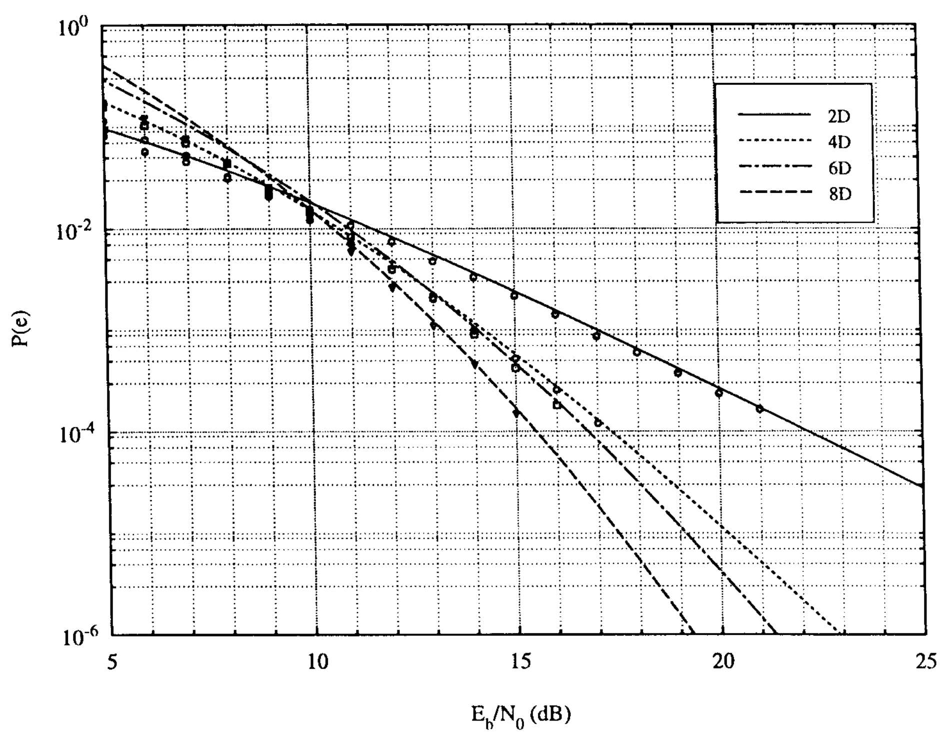


Fig. 2. Error performance of rotated hypercube constellations with $n = 2, 4, 6, 8$ dimensions. Solid curves report the union bound with exact PEP's. Simulation results are also shown.

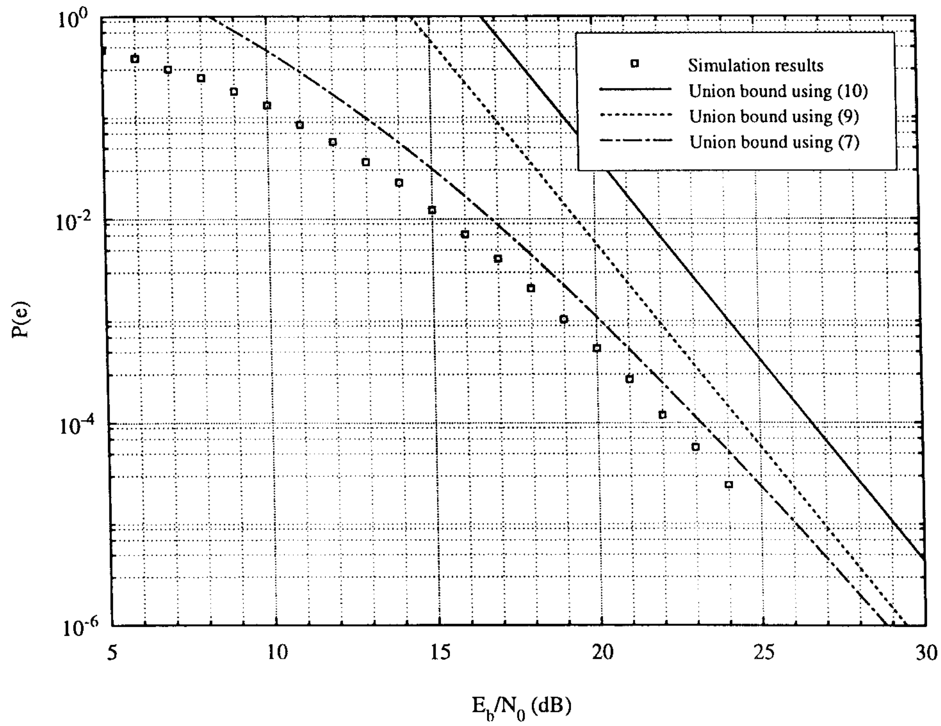


Fig. 3. Error performance of rotated four-dimensional 4-PAM. Upper curves report the union bound obtained by using (10), (9), and (7). Simulation results are also shown.

($n!!$ denotes the *semifactorial* of $n$, namely, $n!! = \prod_{i=0}^{\lfloor n/2 \rfloor}(n-2i)$). The first term in this expression is an upper bound because the series expansion is alternating in signs. It is worth noting that this approximation is already a significant improvement on the corresponding Chernoff bound given by [2]

$$P(\boldsymbol{x} \rightarrow \hat{\boldsymbol{x}}) \leq \frac{1}{\prod_{i=1}^{L}(1+\delta_i^2/4N_0)} = \frac{4^L}{[d_p^{(L)}]^2}N_0^L + O(N_0^{L+1}) \quad (10)$$

asymptotically looser by $(10/L)\,\log_{10}(2^{L+1}L!/(2L-1)!!)$ decibels than the previous, more accurate, result.

## IV. Results and Conclusions

In this section we consider the error performance of two $n$-dimensional signal sets: i) a rotated $n$-dimensional hypercube (corresponding to the set of pints $\{-1, +1\}^n$) with a spectral efficiency of 1 bit/dimension; ii) a rotated $n$-dimensional signal set $\{-3, -1, +1, +3\}^n$ with a spectral efficiency of 2 bits/dimension ($4^n$ points), which can be viewed as the $n$th power of 4-PAM or as a cubic lattice constellation. In both cases, multidimensional rotation matrices are taken from [6].

In the first case, the signal set constellation is geometrically uniform (GU) so that the union bound (2) simplifies to

$$P(e) \leq \sum_{\boldsymbol{x}_0 \neq \hat{\boldsymbol{x}}} P(\boldsymbol{x}_0 \rightarrow \hat{\boldsymbol{x}}). \quad (11)$$

In the other case, we neglect edge effects and approximate the union bound by (11) as well.

Fig. 2 reports performance results in terms of symbol error probability $P(e)$ versus $E_b/N_0$ for constellation i) (the rotated hypercube in two, four, six, and eight dimensions). The diagrams show the union bound obtained with our exact expression of the PEP together with corresponding simulation results. It can be observed that the solid curves (reporting the union bound with exact PEP's) are asymptotically tight with simulation.

Similar results are reported in Fig. 3 for constellation ii). The curves show the union bound (11) evaluated by using a) the Chernoff bound—(10); b) the asymptotic expansion of the exact PEP—(9); c) the exact PEP—(7); and d) simulation results. We observe that even the exact union bound c) is no more asymptotically tight. This is a consequence of the fact that a large number of terms summed up in the union bound are redundant and edge effects have been neglected.

We presented an exact expression of the PEP for component-interleaved multidimensional schemes for the Rayleigh fading channel. The expression is derived by following the approach introduced in [1] applied to the current scenario. This provides better approximation of the error performance than obtained by using the Chernoff bound. The proposed bound can be used for more accurate analysis and design of high-diversity multidimensional constellations for the fading channel.

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# $\mathbf{Z}_{2^k}$-Linear Codes

Claude Carlet

***Abstract*—We introduce a generalization to $\mathbf{Z}_{2^k}$ of the Gray map and generalized versions of Kerdock and Delsarte–Goethals codes.**

***Index Terms*—Galois rings, Gray map, nonlinear codes.**

## I. Introduction

Recently, the Gray map has been extensively used to construct binary codes from codes over $\mathbf{Z}_4 = \mathbf{Z}/4\mathbf{Z}$ (quaternary codes). Recall the definition of this mapping:

*Definition 1:* The Gray map is the mapping $G$ from $\mathbf{Z}_4$ to $\mathrm{GF}(2)^2$ defined by

$$G(0) = (0, 0);\ G(1) = (0, 1);\ G(2) = (1, 1);\ G(3) = (1, 0). \quad (1)$$

$G$ is coordinatewisely extended to a mapping from $(\mathbf{Z}_4)^m$ to $\mathrm{GF}(2)^{2m}$.

The main quality of the Gray map is that it is distance preserving: define the Lee weights of the elements $0, 1, 2,$ and $3$ of $\mathbf{Z}_4$ to be the Hamming weights of $G(0), G(1), G(2),$ and $G(3)$, which are, respectively, $0, 1, 2,$ and $1$; define the Lee weight of a quaternary word to be the sum of the Lee weights of its coordinates. Then, for every quaternary words $u$ and $v$, the Hamming distance between the binary words $G(u)$ and $G(v)$ is equal to the Lee weight of $u - v$ (i.e., the Lee distance between $u$ and $v$), despite the fact that $G(u) + G(v)$ is not equal to $G(u - v)$, in general.

Any element $(z_0, z_1)$ of $\mathrm{GF}(2)^2$ can be identified to the Boolean function: $\epsilon \rightarrow z_\epsilon$, $(\epsilon \in \mathrm{GF}(2))$. Thus the Gray map can be considered as a mapping from $\mathbf{Z}_4$ to the set of Boolean functions on $\mathrm{GF}(2)$, and extended to a mapping from the set of all $\mathbf{Z}_4$-valued functions on a given set $\mathcal{E}$ to the set of all Boolean functions on $\mathcal{E} \times \mathrm{GF}(2)$: any $\mathbf{Z}_4$-valued function $f(x)$ can be written in the form $g(x) + 2h(x)$, where $g$ and $h$ take the values 0 and 1 only; the image of the function $f(x)$ by the Gray map is the Boolean function $(x, \epsilon) \in \mathcal{E} \times \mathrm{GF}(2) \rightarrow h(x) \oplus \epsilon g(x)$, where $g$ and $h$ are considered as valued in $\mathrm{GF}(2)$.

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## II. A Generalization of the Gray Map to $\mathbf{Z}_{2^k}$

There does not exist a distance preserving mapping from $\mathbf{Z}_8$, provided with a translation-invariant distance, to $\mathrm{GF}(2)^3$, provided with the Hamming distance. We give now a distance preserving generalization to $\mathbf{Z}_{2^k} = \mathbf{Z}/2^k\mathbf{Z}$ of the Gray map. Obviously, it cannot be a mapping from $\mathbf{Z}_{2^k}$ to $\mathrm{GF}(2)^k$.

*Definition 2:* Let $k$ be any positive integer, $u$ any element of $\mathbf{Z}_{2^k}$, and $\sum_{i=1}^k 2^{i-1} u_i$ its binary expansion $(u_i = 0, 1)$. The image of $u$ by the generalized Gray map is the following Boolean function on $\mathrm{GF}(2)^{k-1}$

$$G(u)\colon (y_1, \cdots, y_{k-1}) \rightarrow u_k + \sum_{i=1}^{k-1} u_i\, y_i.$$

The generalized Gray map is a mapping from $\mathbf{Z}_{2^k}$ onto the Reed–Muller code of order 1, $\mathrm{R}(1, k-1)$. When $k = 2$, $\mathrm{R}\,(1, 1)$ being equal to the set of all the Boolean functions on $\mathrm{GF}(2)$, we obtain the usual Gray map, which is a mapping from $\mathbf{Z}_4$ to $\mathrm{GF}(2)^2$. In the general case, we can, naturally, identify any Boolean function on $\mathrm{GF}(2)^{k-1}$ to a binary word of length $2^{k-1}$ by listing all its values. We obtain a nonsurjective mapping from $\mathbf{Z}_{2^k}$ to $\mathrm{GF}(2)^{2^{k-1}}$. For instance, when $k = 3$, the images of the elements of $\mathbf{Z}_8$ are the following words of length 4:

$$G(0) = (0, 0, 0, 0);\ G(1) = (0, 1, 0, 1);\ G(2) = (0, 0, 1, 1);$$
$$G(3) = (0, 1, 1, 0);\ G(4) = (1, 1, 1, 1);\ G(5) = (1, 0, 1, 0);$$
$$G(6) = (1, 1, 0, 0);\ G(7) = (1, 0, 0, 1);$$

and we obtain all the words of even weights. This is no more the case for $k > 3$. We extend the generalized Gray map, coordinatewisely, into a mapping from $(\mathbf{Z}_{2^k})^n$ to ${F_2}^{2^{k-1}n}$. As in the case of $\mathbf{Z}_4$, it can also be extended to all $\mathbf{Z}_{2^k}$-valued functions: let $\mathcal{E}$ be any set and $f$ any function from $\mathcal{E}$ to $\mathbf{Z}_{2^k}$; let $\sum_{i=1}^k 2^{i-1} f_i(x)$ be its binary expansion ($f_i$, $i = 1, \cdots, k$ are Boolean functions on $\mathcal{E}$). The image of $f$ by the generalized Gray map is the Boolean function on $\mathcal{E} \times GF(2)^{k-1}$

$$G(f)\colon (x, y_1, \cdots, y_{k-1}) \rightarrow f_k(x) + \sum_{i=1}^{k-1} f_i(x)\, y_i.$$

The reverse image of Hamming distance by the generalized Gray map is a translation-invariant distance:

*Proposition 1:* Let $u$ and $v$ be two elements of $\mathbf{Z}_{2^k}$. The Hamming distance between $G(u)$ and $G(v)$ is equal to the Hamming weight of $G(u - v)$.

*Proof:* Consider the following three cases: $u = v$, $u = v + 2^{k-1}$, and $u \not\equiv v \bmod 2^{k-1}$. When $u = v$ (respectively, $u = v + 2^{k-1}$), we have $G(u - v) = G(u) + G(v)$; when $u \not\equiv v \bmod 2^{k-1}$, the distance between $G(u)$ and $G(v)$ and the weight of $G(u - v)$ are both equal to $2^{k-2}$, since $G(u) + G(v)$ and $G(u - v)$ are nonconstant affine functions. □

The Gray map has another nice property: the relationship between the weight of the image $G(f)$ of a $\mathbf{Z}_4$-valued function $f$ on a set $\mathcal{E}$ and exponential sums involving $f$ and $-f$

$$w(G(f)) = |\mathcal{E}| - \tfrac{1}{2}\left(\sum_{x \in \mathcal{E}} \left(i^{f(x)} + i^{-f(x)}\right)\right). \quad (2)$$