# Cross-Packing Lattices for the Rician Fading Channel

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***Abstract*—We introduce cross-packing lattices for Rician fading channels, motivated by a geometric interpretation stemming from the pairwise error probability analysis. We approximate the star bodies arising from the pairwise error probability analysis with $n$-dimensional crosses of radius $t$, consisting of $2nt + 1$ unit cubes, for some positive integer $t$. We give a construction for a family of cross-packing lattices for all dimensions and any minimum cross distance $2t+1$. We show by simulations how our new cross-packing lattices perform compared to other known lattices over the Rician fading channel, for different values of the Rician $K$-factor.**

## I. Introduction

In recent years, lattice signaling has been shown to provide an excellent tradeoff between performance and implementation complexity. Finite sets of lattice points are known as lattice codes and are used as signals. Most literature on the design of lattices has focused on the Gaussian channel [5], [18] and the unconstrained Gaussian channel [6], [12], [13], [14], [15]. Furthermore, the design of lattices matched to a given Rayleigh fading channel was studied in [1], [2], [3], [8], [9]. For the Gaussian channel, finding a good lattice code translates into a sphere packing problem with respect to the Euclidean distance [4], whereas for the Rayleigh fading channel, it translates into the design of full-diversity lattices with non-vanishing minimum product distance [1], [2].

In this work, we design integer lattices matched to the Rician fading channel. It is known from [7], [8] that, for certain parameters, the set of points, for which the pairwise decoding error probabilities are below a given threshold, form star bodies. A star body $\mathcal{S} \subseteq \mathbb{R}^n$ is a set containing a point $p_0$ so that for each point $p \in \mathcal{S}$, the line segment $p_0p$ lies in $\mathcal{S}$. We will approximate these star bodies in $n-$space with $n$-dimensional crosses (also called *cross polyominoes* [10]) of radius $t$. Such a cross can be constructed by an $n$-dimensional unit cube as the center, where all of its $2n$ faces are extended by $t$ unit cubes. Since it was shown in [17] that packings of such crosses correspond to sets of points with a given minimum cross distance, we derive cross-packing lattices with respect to this cross distance and compare their performance to other known lattices in simulations.

The paper is structured as follows. In Section II, we recall some definitions and properties of lattices. Then we review the setting of a memoryless Rician fading channel and an upper bound for the pairwise error probability. We plot the points in $2-$space, whose pairwise error probability with the origin are below a given threshold. These regions have different shapes, namely spherical or star-shaped with respect to different Rician $K$-factors at different volume-to-noise ratios (VNR). In Section III we derive so-called cross-packing lattice. We give a general construction for lattices in real $n-$space, depending on an integer variable $t$, and prove that these lattices have minimum cross distance $2t + 1$ for any positive integer $t$. In Section IV, we show by simulations how these cross-packing lattices perform compared to other lattices under Rician fading. We finally give concluding remarks in Section V.

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**Notation.** The sets $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ denote the set of natural numbers, ring of integers, field of rational numbers, and the field of real numbers, respectively. Let $\mathbf{v} \in \mathbb{R}^n$ be a vector, then its $j$-th entry is denoted by $v_j$. Its Euclidean norm is denoted by $\|\mathbf{v}\|$. The $i$-th unit vector of $\mathbb{R}^n$ is denoted by $\mathbf{e}_i$.

## II. Lattices and the Rician Fading Channel

We first recall the notion of lattice, which is essential throughout the paper. An $n$-dimensional *lattice* $\Lambda$ is the set of points $\{\mathbf{x} = \mathbf{uB} | \mathbf{u} \in \mathbb{Z}^n\}$, whose *generator matrix* $\mathbf{B} = \left(\mathbf{b}_1^T, \ldots, \mathbf{b}_n^T\right)^T$ is formed by stacking the $n$-dimensional row vectors $\mathbf{b}_1, \ldots, \mathbf{b}_n \in \mathbb{R}^n$. The volume of a lattice is defined as

$$\mathrm{vol}(\Lambda) \triangleq \det(\mathbf{B}).$$

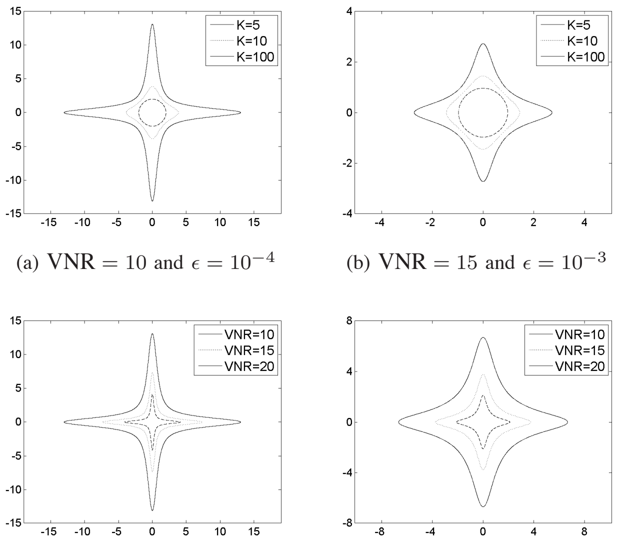
By $d_{\min}^E(\Lambda)$ we denote the minimum Euclidean distance of a lattice $\Lambda$. The lattice $\Lambda$ is called an *integer lattice*, if $\Lambda \subseteq \mathbb{Z}^n$.

We now look at a *memoryless Rician fading channel* with ideal channel state information (CSI) at the receiver and no CSI at the transmitter. In this setting the transmitted signal arrives at the receiver in several parallel paths and hence exhibits multi-path interference. The ratio between the power in the direct path and the power in the scattered paths is given by the parameter $K$. The received signal amplitude is then Rice distributed [7], *i.e.*, if $\mathbf{x} = (x_1, \ldots, x_n) \in \Lambda$ is transmitted, the corresponding channel output is $\mathbf{y} = (y_1, \ldots, y_n)$ with each component given by

$$y_i = a_i x_i + z_i, \tag{1}$$

where $a_i$ is the normalized fading amplitude for the $i$-th component. For Rician fading this has the probability density function

$$f(a) = 2a(1+K) \times \exp\left(-K - a^2(1+K)\right) \times I_0\left(2a\sqrt{(K(K+1))}\right), \tag{2}$$




(a) VNR $= 10$ and $\epsilon = 10^{-4}$ (b) VNR $= 15$ and $\epsilon = 10^{-3}$

(c) $K = 5$ and $\epsilon = 10^{-4}$ (d) $K = 10$ and $\epsilon = 10^{-5}$

Fig. 1: Points $\mathbf{x} \in \mathbb{R}^{\mathbf{2}}$ with (3) $\leq \epsilon$ for varying VNR (in dB) and $K$.

where $I_0$ is the zero-order modified Bessel function of the first kind. Note that $z_i$ in (1) is white Gaussian noise $\sim \mathcal{N}(0, \sigma^2)$, for $1 \leq i \leq n$. Since the channel state information is available at the receiver, the maximum-likelihood (ML) decoding criterion is to solve the following minimization problem

$$\text{argmin} \left\{ \sum_{i=1}^{n} (y_i - a_i x_i)^2 \mid \mathbf{x} \in \Lambda \right\}.$$

For our simulations we normalize the volume of all lattices to 1 and define the *volume-to-noise* ratio as

$$\text{VNR} = \frac{\text{vol}(\Lambda)^{2/n}}{8\sigma^2} = \frac{1}{8\sigma^2}.$$

In Section IV, we evaluate the performance of lattices as a function of noise variance. A lattice performs better than another if it attains a smaller error probability for a fixed VNR.

We now investigate the behavior of the pairwise error probability around the origin (*i.e.* the zero lattice point). Because of the linearity of lattices, the behavior is the same for all lattice points. The pairwise error probability for the origin can be upper-bounded by [7]:

$$\Pr(\mathbf{0} \to \mathbf{x}) \leq \prod_{i=1}^{n} \left( \frac{K+1}{K+1+\text{VNR} \cdot x_i^2} \times \exp\left( -\frac{K \cdot \text{VNR} \cdot x_i^2}{K+1+\text{VNR} \cdot x_i^2} \right) \right), \quad (3)$$

Note that we reformulated the above formula in terms of volume-to-noise ratio, replacing $\frac{E_s}{8\sigma^2}$ by VNR, where $E_s$ is the average symbol energy of a finite lattice constellation, as defined in [7].

As in [8], [9], Figure 1 depicts the regions of points in $\mathbb{R}^2$, for which the right hand side of (3) is at most $\epsilon$. In Figures 1(a) and 1(b), one sees that for $K = 5$ these regions are star-shaped, while for $K = 10$ they resemble squares, and for $K = 100$ they turn into spheres. In Figures 1(c) and 1(d) we observe that for $K = 5, 10$ changing VNR from 10dB to 20dB scales but does not change the star-shape for the regions.

In this paper, we will focus on these star-shaped spheres and construct matching lattices for this setting. To do so, we will approximate the $n$-dimensional star bodies as $n$-dimensional crosses and construct so-called integer *cross-packing lattices*, *i.e.* lattices that give rise to good (but not perfect) cross packings, in the following section.

## III. Cross-packing lattices

In this section we will construct lattices that give rise to good cross packings in $\mathbb{Z}^n$. More formally, we construct lattices with a given minimum cross distance. This *cross distance* was defined in [17] as

**Definition 1.** For every $\mathbf{x}, \mathbf{y} \in \mathbb{Z}^n$, we define

$$d_+(\mathbf{x}, \mathbf{y}) \triangleq \begin{cases} |x_i - y_i| & \text{if } x_i \neq y_i \text{ and } x_j = y_j \forall j \neq i, \\ 0 & \text{if } \mathbf{x} = \mathbf{y}, \\ \infty & \text{if } \exists i, j : i \neq j, x_i \neq y_i, x_j \neq y_j. \end{cases}$$

Note that the cross distance is only defined for integer vectors, hence the notion of a *normalized* cross distance is impractical.

In [17] it was also shown that any set of points in $\mathbb{Z}^n$ is a packing of crosses of radius $t$ if the minimum cross distance of the set is at least $2t+1$. Using Construction A, any known construction for finite codes with minimum cross distance $2t+1$ can be used to construct cross-packing lattices. For $t = 1$ there exists a perfect packing for any $n$ by the Golomb-Welch construction [10]. For $t = 2, 3$ and some specific values of $n$ the perfect $t$-shift $n$-designs from [11], [16] give rise to good cross packings too.

In this paper however, we will give a general construction for cross-packing lattices with arbitrary radius $t$ for any dimension $n$. The codes from [17] are defined for general $t$ and $n$ as linear codes over $\mathbb{Z}_{2^m}$ for $m \in \mathbb{N}$. Nonetheless, the respective lattices, arising from Construction A, give rise to sparser cross-packings in $\mathbb{Z}^n$ than the lattices given by the following new cross-packing lattice construction.

**Construction I (cross-packing lattices).** Iteratively define the following lattice bases (row wise):

$$\mathbf{B}_2 \triangleq \begin{pmatrix} 1 & t+2 \\ t+1 & 1 \end{pmatrix},$$

and

$$\mathbf{B}_3 \triangleq \begin{pmatrix} 1 & t^2+t+1 & 0 \\ t+1 & -1 & 0 \\ 0 & t+1 & -1 \end{pmatrix},$$

and for $n \geq 4$

$$\mathbf{B}_n \triangleq \left( \begin{array}{ccc|c} & & & 0 \\ & \mathbf{B}_{n-1} & & \vdots \\ & & & 0 \\ \hline 0 & \dots & 0 \quad t+1 & -1 \end{array} \right).$$

We now investigate the minimum cross distance of the constructed lattices $\Lambda_n$ with generator matrices $\mathbf{B}_n$. To do so we need the following lemmas, and the notion of *Hamming weight*, which is the number of non-zero coordinates of a given vector.

**Lemma 2.** *For $n \geq 3$ the codewords of Hamming weight 1 of the cross-packing lattice $\Lambda_n$ are multiples of*

$$v\mathbf{e}_i, \quad 1 \leq i \leq n$$

*with $v = (t+1)(t^2+t+1)+1 = t^3+2t^2+2t+2$.*

*Proof:* For $n = 3$ this can easily be checked. Inductively the weight-1 words of $\Lambda_{n+1}$ are the ones of $\Lambda_n$ (with a zero appended), plus a new one whose non-zero entry is in the last position. By the shape of the basis, this new vector can only be a linear combination of the last row of $\mathbf{B}_{n+1}$ and the weight-1 word $(0 \ldots \ 0 \ v \ 0)$, *i.e.*

$$\lambda(0 \ldots 0, \ t+1, \ -1) + \mu(0 \ldots 0, \ v, \ 0).$$

To make the second last coordinate zero, one needs $\lambda = -\mu v/(t+1)$. Since $t+1$ does not divide $v$ for any $t \geq 1$, we get that the integer solution of the smallest absolute value is $\lambda = -v$ (and $\mu = t+1$), thus the new word is a multiple of $(0 \ldots \ 0, \ v) = v\mathbf{e}_{n+1}$. ■

**Lemma 3.** *For $n \geq 3$ the codewords $\mathbf{x} = (x_1, \ldots, x_n)$ of the cross-packing lattice $\Lambda_n$ with $x_1 = x_2 = \cdots = x_{n-2} = 0$ and $|x_{n-1}| \leq t$ have $|x_n| \geq t^2+t+1$.*

*Proof:* For $n = 3$, since $x_1 = 0$, we get $\mathbf{x} = \lambda(0, \ v, \ 0) + \mu(0, \ t+1, \ -1)$ with $v = (t+1)(t^2+t+1)+1$ (see Lemma 2). If $|x_2| \leq t$, we get $\lambda \neq 0$ and

$$-t \leq \lambda v + \mu(t+1) \leq t \iff \frac{-t-\lambda v}{t+1} \leq \mu \leq \frac{t-\lambda v}{t+1}.$$

We distinguish between three cases:

- If $\lambda \geq 1$, then
$$\mu \leq \frac{\lambda - t}{t+1} - \lambda(t^2+t+1) \leq -(t^2+t+1).$$
- If $\lambda \leq -2$, then
$$\mu \geq -\frac{\lambda+t}{t+1} - \lambda(t^2+t+1) \geq t^2+t+1.$$
- If $\lambda = -1$, then
$$\mu \geq -\frac{t-1}{t+1} + (t^2+t+1) \geq t^2+t+\frac{2}{t+1}.$$
  Since $\mu \in \mathbb{Z}$, we get $\mu \geq t^2+t+1$.

Since $x_3 = -\mu$, the statement follows for $n = 3$. Then one can inductively use the same procedure to prove the statement for general $n \geq 3$ (using the fact from Lemma 2, that all elements of Hamming weight 1 have the non-zero entry $v$). ■

**Lemma 4.** *For $n \geq 3$ the codewords $\mathbf{x} = (x_1, \ldots, x_n)$ of the cross-packing lattice $\Lambda_n$ with Hamming weight 2, and $x_j, x_n \neq 0$ for some $j \in \{1, \ldots, n-1\}$, such that $|x_j| \leq t$, either have $|x_n| \geq t^2+t+1$, or $x_n$ is a multiple of $t^2+1$.*

*Proof:* For $n = 3$, if $x_1 = 0$, this is proven by Lemma 3. If $x_2 = 0$, then $\mathbf{x} = \lambda(1, \ t^2+t+1, \ 0) + \mu(t+1, \ -1, \ 0) + \gamma(0, \ t+1, \ -1)$, thus $\lambda(t^2+t+1) - \mu + \gamma(t+1) = 0$, *i.e.*

$$\gamma(t+1) = \mu - \lambda(t^2+t+1). \tag{4}$$

If $|x_1| \leq t$, we get $-t \leq \lambda + \mu(t+1) \leq t$ or equivalently,

$$-t - \mu(t+1) \leq \lambda \leq t - \mu(t+1). \tag{5}$$

Together we get that

$$\mu(t^2+t+1) - t^2 - \frac{t-\mu}{t+1} \leq \gamma \leq \mu(t^2+t+1) + t^2 + \frac{t+\mu}{t+1},$$

which implies that $x_3 = -\gamma \geq t^2+t+1$ for $\mu \leq -2$ or $x_3 = -\gamma \leq -(t^2+t+1)$ for $\mu \geq 2$. For $\mu = -1$ we get that

$$\gamma = -\lambda t - \frac{\lambda+1}{t+1}.$$

Since $\gamma \in \mathbb{Z}$, then with the restrictions on $\lambda$ in (5) we have that $\lambda \in [1, 2t+1]$. However, because $t+1$ should divide $\lambda+1$, we get $\lambda \in \{t, 2t+1\}$. For the first case, one gets $\gamma = -(t^2+1)$, and hence $x_3$ is a multiple of $t^2+1$. For the second $\gamma = -(2t^2+t+2)$, which means $|x_3| > t^2+t+1$. The analogous results hold for $\mu = 1$.

For $n \geq 4$, $\mathbf{x} \in \Lambda_n$ of weight 2 is a linear combination of the last row of $\mathbf{B}_n$ and a weight-2 vector $(\ldots w_1 \ldots w_2 \ 0) \in \Lambda_{n-1}$. We write this as:

$$\mathbf{x} = \lambda(\ldots w_1 \ldots w_2, \ 0) + \mu(0 \ldots 0, \ t+1, \ -1).$$

If $x_{n-1} = 0$, then $\lambda w_2 = -\mu(t+1)$, and if $-t \leq \lambda w_1 \leq t$ (otherwise $|x_j| > t$), then

$$-\frac{t}{t+1} \leq \mu \frac{w_1}{w_2} \leq \frac{t}{t+1}.$$

Inductively, we know that either $|w_2| \geq t^2+t+1$ or $|w_2| = \alpha(t^2+1)$ for some $\alpha \in \mathbb{Z}$. The first case is analogous to the proof of Lemma 3. For the second case, note that $t+1$ does not divide $t^2+1$. Therefore,

$$\mu = -\frac{\lambda w_2}{t+1} = \pm \frac{\lambda\alpha(t^2+1)}{t+1}$$

is an integer if and only if $\lambda\alpha$ is a multiple of $t+1$, hence $\mu$ is a multiple of $t^2+1$. ■

**Theorem 5.** *The lattices $\Lambda_n \subseteq \mathbb{Z}^n$ with basis $\mathbf{B}_n$ constructed according to Construction I have minimum cross distance $2t+1$.*

*Proof:* As shown in [17], it is enough to check that all lattice points on any of the axis (*i.e.* the lattice codewords of Hamming weight 1) have squared Euclidean distance at least $2t+1$ and that all points with two non-zero coordinates (*i.e.* the codewords with Hamming weight 2) have one coordinate of absolute value at least $t+1$.

For $n = 2$ one can easily check that the points on the $x$-axis are multiples of $(t^2+3t+1, \ 0)$ and on the $y$-axis they are multiples of $(0, \ t^2+3t+1)$. One sees that $t^2+3t+1 \geq 2t+1$. Moreover, all other points have at least one coordinate of absolute value at least $t+1$. This can be seen as follows:

If $\lambda(1,\ t+2)+\mu(t+1,\ 1)$ is such that both coordinates have absolute value at most $t$, then

$$-t \leq \lambda+\mu(t+1) \leq t \iff -\mu(t+1)-t \leq \lambda \leq t-\mu(t+1),$$

$$-t \leq \lambda(t+2)+\mu \leq t \iff -\frac{\mu+t}{t+2}-t \leq \lambda \leq \frac{t-\mu}{t+2}.$$

The only integer solution that fulfills all of the above inequalities is $(\lambda, \mu)=(0,0)$.

For $n=3$ we distinguish the three different cases where two coordinates are zero at the same time. Analogously to the two-dimensional case one gets that those points are multiples of $(0,\ 0,\ v), (0,\ v,\ 0)$ and $(v,\ 0,\ 0)$ with $v=t^3+2t^2+2t+2$, respectively. Similarly to the 2-dimensional case one can check that the points of Hamming weight 2 have at least one coordinate of absolute value at least $t+1$, see Lemma 4.

For $n \geq 4$, we want to show that $\mathbf{B}_n$ is a basis of a lattice with minimum cross distance $2t+1$. Note that any linear combination not involving the last row of $\mathbf{B}_n$ is a point of $\Lambda_{n-1}$ and thus valid by induction. Therefore we now focus on the linear combinations $\mathbf{x}=\lambda(\mathbf{y},\ 0)+\mu\mathbf{b}_n$ for $\lambda, \mu \in \mathbb{Z}\backslash\{0\}$, where $\mathbf{y} \in \Lambda_{n-1}$ and $\mathbf{b}_n$ is the last row of $\mathbf{B}_n$.

Case 1: If $\mathbf{y}$ is of Hamming weight at least 3, then $\mathbf{x}$ has Hamming weight at least 3, as well.

Case 2: If $\mathbf{y}$ has Hamming weight 2 and $y_{n-1}=0$, then $\mathbf{x}$ has Hamming weight 4. If $y_{n-1} \neq 0$, then either $\mathbf{x}$ has Hamming weight 3, or, by Lemma 4, $\mathbf{x}$ has at least one coordinate with absolute value at least $t+1$.

Case 3: If $\mathbf{y}$ has Hamming weight 1 and $y_{n-1}=0$, then $\mathbf{x}$ has Hamming weight 3. Now assume that $y_{n-1} \neq 0$. We know from Lemma 2 that $(\mathbf{y},\ 0)=(0\ldots0,\ (t+1)^2t+(t+2),\ 0)$. Hence, either $\mathbf{x}=(0\ldots0,\ (t+1)^2t+(t+2))$ or $\mathbf{x}$ has Hamming weight 2. In the latter case, we know from Lemma 3 that at least one of the coordinates has absolute value at least $t+1$.

Case 4: If $\mathbf{y}=\mathbf{0}$, then $\mathbf{x}$ has Hamming weight 2 and $|x_{n-1}| \geq t+1$.

Overall, it follows that $\Lambda_n$ has cross distance at least $2t+1$. ■

**Remark 6.** In general the cross-packing lattices $\Lambda_n$ do not give rise to perfect packings. However, the lattices with basis $\mathbf{B}_2, \mathbf{B}_3$ are perfect cross packings for $t=1$. In fact, these two lattices correspond to the construction of [10].

## IV. SIMULATIONS

In this section we compare the performance of our cross-packing lattices with other known lattices by simulations. We use the cross-packing lattices defined in Section III for $t=1,2$. For $n=2$ we compare the performance of the cross-packing lattices to the one of the hexagonal lattice [4] and the algebraic rotation (AR) lattice arising from the algebraic number field $\mathbb{Q}((1+\sqrt{5})/2)$ [1]. The former is known to be optimal for non-fading additive white Gaussian noise channels [4], while the latter is the best known cubic lattice code for Rayleigh fading channels for high VNR [1]. The simulation results are depicted in Figure 2.

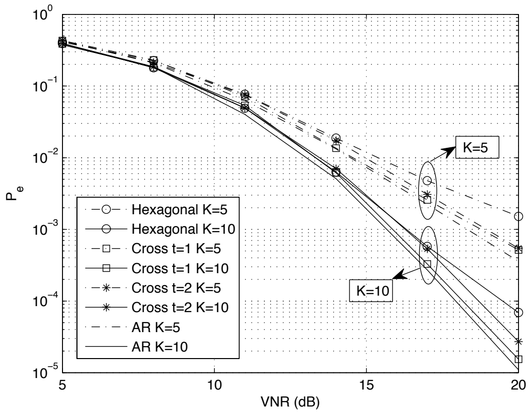



Fig. 2: Lattices in $\mathbb{R}^2$ for $K=5$ and $K=10$.

One can see that for high VNR the cross-packing lattices outperform the hexagonal lattice. We also observed that these cross-packing lattices outperform the integer lattice $\mathbb{Z}^2$ (not included in the figure). In the VNR region simulated in the figures, cross-packing lattices for $t>2$ perform worse than the ones for $t=1,2$, which is why we refrained from including them in the plots.

For $n=8$ we compare the cross-packing lattices to the 8-dimensional Gosset lattice $E_8$ [4], and AR lattices [1]. The simulation results are presented in Figure 3.

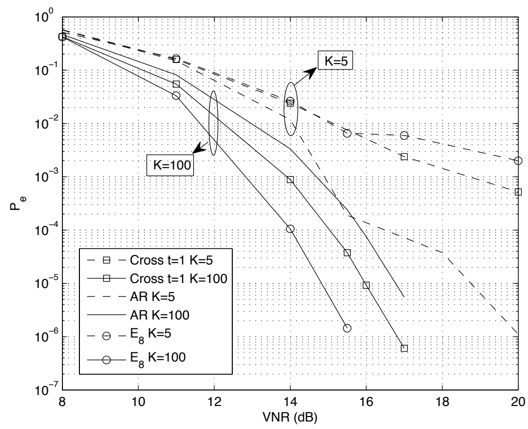



Fig. 3: Lattices in $\mathbb{R}^8$ for $K=5$ and $K=100$.

In this setting the Gosset lattice $E_8$ outperforms all the other lattices for $K=100$. Our cross-packing lattices with $t=1,2$ perform better than the algebraic rotation lattice and the unit lattices. This makes our cross-packing lattice constructions suitable for higher $K$ in comparison to AR lattices, which matches our observations with regard to the pairwise error probability regions in Figure 1.

Finally, we compare the error performance of our cross-packing lattices with the perfect packings of crosses of radius

$t = 1$, arising from the Golomb-Welch construction [10], in Figure 4.

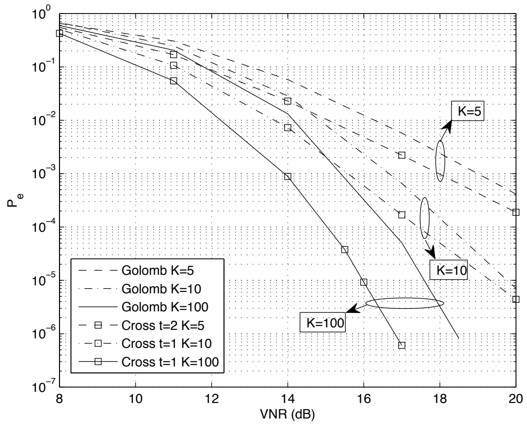



Fig. 4: Lattices in $\mathbb{R}^8$ for $K = 5$, $K = 10$, and $K = 100$.

One can see, that our cross-packing lattices outperform the Golomb-Welch lattices for various values of $K$.

For small $K$, the Rician channel behaves similarly to the Rayleigh fading channel. This explains why for $K = 5$ and $K = 10$ the lattices from algebraic rotations perform the best. For larger $K$, the Rician channel behaves like the AWGN channel and Euclidean distance plays the main role in this region. In these regimes, we expect algebraic rotation lattices to perform similar to $\mathbb{Z}^n$, which explains why they perform worse than our cross-packing lattices. The same reasoning explains why the hexagonal lattice is outperformed for $K = 5, 10$. For $K = 100$ (not included in figures), due to the similarity to the AWGN channel, the hexagonal lattice outperforms the other depicted lattices. The simulations for other values of $K \leq 100$ show similar results: the performance of our lattices is between the one of AR lattices and the one of good sphere-packing lattices.

Overall, our cross-packing lattices perform well for a broad range of values for $n$, $K$, and VNR. For a fixed set of parameters they are outperformed by other known lattices, but none of these other lattices are suitable for a broad range of parameters. Moreover, cross-packing lattices are integer lattices, while algebraic rotation lattices are not. Thus, the marginal coding loss for small $K$ is traded off with the simplicity of having integer components in our cross-packing lattice constructions.

## V. Conclusion

In this work, we designed lattices for the Rician fading channel. Due to the similarity of star-shaped pairwise error probability regions to $n$-dimensional crosses, we derive a construction for $n$-dimensional cross-packing lattices, *i.e.* lattices that give rise to good cross packings, for any $n \geq 2$. We simulate the performance of these cross-packing lattices and compare the error performance results to other known lattices including the hexagonal lattice, the Gosset lattice $E_8$, and algebraic rotation lattices.

On one hand, our cross-packing lattices outperform the hexagonal lattice for small $K$, but perform worse than algebraic rotation lattices. However, the gain of the algebraic rotation lattices is marginal and these lattices have the disadvantage of being non-integer, whereas our cross-packing lattices are defined in $\mathbb{Z}^n$. On the other hand, for larger $K$, our cross-packing lattices beat algebraic rotation lattices. In these regimes, the best lattices are the ones with better coding gain (center density) with respect to Euclidean distance.

Overall, we showed that cross-packing lattices, although not optimal for a fixed set of parameters $n$, $K$, and VNR, perform well when compared in a broad range of parameters.

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