XY Precoder for MIMO Systems

Shuiyin Liu, Yi Hong, and Emanuele Viterbo
ECSE Department, Monash University
Melbourne, VIC 3800, Australia
Email: shuiyin.liu, yi.hong, emanuele.viterbo@monash.edu

***Abstract*—In multiple-input multiple-output (MIMO) channels with discrete input alphabets, at high signal-to-noise ratio (SNR), maximizing the minimum Euclidean distance ($d_{\min}$) between all possible received constellation points is known to be the optimal precoding strategy. However, finding the optimal precoder has been proved to be NP-hard. For large MIMO, a promising practical approach is to transform the channel into parallel 2×2 MIMO subchannels and then precode each of them separately. However, existing methods are mostly based on heuristic subchannel pairing schemes and require numerical search/optimization in the design phase. In this work, we propose a novel real-valued precoder, named as *XY-precoder*, which enjoys an explicit construction, a provable $d_{\min}$, a provably optimal subchannel pairing scheme, and low ML-decoding complexity. We prove that the XY-precoder achieves the same diversity order as the best known precoder, but with a much lower decoding complexity. Simulation results confirm that the error performance of XY-precoder is almost the same as that of the best known precoders.**

I. INTRODUCTION

The problem of constructing precoders for multiple-input multiple-output (MIMO) systems with discrete-constellation inputs (e.g. M-QAM) has received attention in the past decade. Assuming the channel state information (CSI) is known at the transmitter, various design criteria for linear precoding have been proposed in the literature, for example, minimizing the mean-square error (MSE) [1], maximizing the received signal-to-noise ratio (SNR) [2], maximizing the minimum singular value [3], or maximizing the mutual information [4]. It is known that at high SNR, a *universally optimal precoder* is the one that maximizes the minimum Euclidean distance between any pair of received constellation points [5], commonly referred to as $\max - d_{\min}$ *precoder*. However, the problem of finding the optimal precoder has been proved to be NP-hard [6]. Existing $\max - d_{\min}$ based approaches are mostly limited to 2-dimensional precoding with a low order QAM modulation [7,8].

For high-dimensional precoding, a promising suboptimal approach has been proposed in [9]. The idea is to decompose the eigen-channel matrix into N_{T} subchannels (N_{T}: number of transmit antennas) and pair every two subchannels to yield a set of 2×2 subsystems under certain pairing rule (N_{T} even). Then, the 2-dimensional precoder in [7] is directly applied to each subsystem. The last step is to equalize the minimum distances among all the subsystems by power allocation. This method is therefore referred as to *equal* $d_{\min}$ *precoder* ($\mathrm{E} - d_{\min}$), and has been shown to significantly reduce the bit-error rate (BER) in comparison with other traditional precoding strategies in [1–3]. However, inherited from [7], $\mathrm{E} - d_{\min}$ precoder is constructed by numerical optimization. Another issue is related to the decoding complexity: the precoding matrices for each subsystem are complex valued. This results in the maximum likelihood (ML) detection for each subsystem over a 4-dimension search space. With M-QAM, the corresponding complexity of an exhaustive search ML is proportional to M^2.

From both theoretical and practical perspective, it is always desirable to use a precoder with an explicit construction, provable performance (e.g., achievable $d_{\min}$, optimality of subchannel pairing), and low ML-decoding complexity. Real-valued $\mathrm{E} - d_{\min}$ precoders, called X*-precoder* and Y*-precoder*, have been proposed to reduce the order of exhaustive ML decoding complexity to M in [10]. X-precoder is designed for well-conditioned channel matrices, and Y-precoder is constructed for ill-conditioned channel matrices. However, over a MIMO Rayleigh fading channel, neither X- nor Y-precoder is as good as the original $\mathrm{E} - d_{\min}$ precoder in [9] in terms of BER. Specifically, X-precoder uses a QAM constellation and requires numerical search in the design phase. In contrast, Y-precoder uses a non-QAM constellation and has an explicit construction. These differences make it difficult to combine the constructions of X- and Y-precoder.

The main contribution of this paper is to propose a novel real-valued $\mathrm{E} - d_{\min}$ precoder, named as *XY-precoder*, with an explicit construction, provable performance, and low ML-decoding complexity. The BER performance of XY-precoder is very close to the best known ones in [8,9], and what's more, it enjoys the following benefits:

1) a provable minimum distance
2) valid for any QAM modulation
3) a provably optimal subchannel pairing scheme
4) achieving the same diversity order as that in [8,9]
5) exhaustive ML decoding complexity in the order of M

The key idea is to jointly design a pair of new X- and Y-precoders, in such a way that both of them use the same optimization strategies, i.e., power allocation and subchannel pairings. This new construction criterion allows us to adaptively switch X- and Y-precoder for each subsystem, by simply choosing the one with larger minimum distance (thus referred

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as to XY-precoder). In other words, the proposed XY-precoder takes full advantage of both X- and Y-precoder, thus provides good performance over MIMO Rayleigh fading channels.

Section II presents the system model. Section III describes the construction of XY-precoder. Section IV shows the simulation results and comparisons with other precoders. Section V sets out the theoretical and practical conclusions. The Appendix contains the proofs of the theorems.

II. System Model and Problem Statement

A. MIMO Channel Model with a QAM Constellation

We consider a MIMO wireless system that includes a transmitter and a receiver, with N_{T} and N_{R} antennas ($N_{\mathrm{R}} \geq N_{\mathrm{T}}$), respectively. Assuming the perfect channel state information (CSI) is available at both the transmitter and receiver sides, a precoding matrix $\mathbf{F} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}$ is employed to improve the system performance. Let $\mathbf{x} = [x_1, \cdots, x_{N_{\mathrm{T}}}]^T$ be an $N_{\mathrm{T}} \times 1$ vector of symbols to be transmitted over the MIMO channel. Each symbol x_i is independently chosen from a $M-$QAM constellation $\mathcal{X}$, given by

$$\mathcal{X} = \frac{1}{\sqrt{K}} \left\{ \pm u \pm vj | u, v \in \left\{ 1, 3, \cdots, \sqrt{M}-1 \right\} \right\}, \quad (1)$$

where $K = (2/3)(M-1)$.

The received symbol vector can then be expressed as

$$\mathbf{y} = \mathbf{HFx} + \mathbf{n}, \quad (2)$$

where $\mathbf{F} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}$ is the precoding matrix, $\mathbf{H} \in \mathbb{C}^{N_{\mathrm{R}} \times N_{\mathrm{T}}}$ represents the channel matrix and $\mathbf{n} \in \mathbb{C}^{N_{\mathrm{R}} \times 1}$ is the white Gaussian noise (AWGN) vector with i.i.d. entries $\sim \mathcal{N}_{\mathbb{C}}(0, \sigma_{\mathrm{B}}^2)$. We consider MIMO Rayleigh-fading channels, i.e., $\mathbf{H}$ has i.i.d. entries $\sim \mathcal{N}_{\mathbb{C}}(0,\ 1)$. The precoding matrix $\mathbf{F}$ has a power constraint:

$$\mathrm{trace}\left\{ \mathbf{F}^H \mathbf{F} \right\} = \|\mathbf{F}\|_{\mathrm{F}}^2 = E_{\mathrm{T}}, \quad (3)$$

where $\|\cdot\|_{\mathrm{F}}$ denotes the Frobenius norm of a matrix and E_{T} represents the average transmit power.

B. Max-Minimum Distance Precoding ($\max - d_{\min}$)

At high SNR, a *universally optimal precoder*, in the sense of both the sum-rate maximization and the detection error probability minimization, is the one that maximizes the minimum Euclidean distance between all possible received constellation points [5]. This criterion can be formulated as:

$$\mathbf{F}_{\max - d_{\min}} = \arg \max_{\mathbf{F} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}} d_{\min}(\mathbf{HF}), \text{ s.t. } \|\mathbf{F}\|_{\mathrm{F}}^2 = E_{\mathrm{T}}, \quad (4)$$

where

$$d_{\min}(\mathbf{HF}) \triangleq \min_{\mathbf{x}_i, \mathbf{x}_j \in \mathcal{X}^{N_{\mathrm{T}}}, \mathbf{x}_i \neq \mathbf{x}_j} \left\| \mathbf{HF}(\mathbf{x}_i - \mathbf{x}_j) \right\| = \min_{\mathbf{s} \in \mathcal{S}^{N_{\mathrm{T}}}} \|\mathbf{HFs}\|. \quad (5)$$

Each symbol s_i in the difference vector $\mathbf{s} = [s_1, \cdots, s_{N_{\mathrm{T}}}]^T$ belongs to a finite constellation $\mathcal{S}$, given by

$$\mathcal{S} = \frac{2}{\sqrt{K}} \left\{ \pm \hat{u} \pm \hat{v}j | \hat{u}, \hat{v} \in \left\{ 0, 1, \cdots, \sqrt{M}-1 \right\} \right\}. \quad (6)$$

We refer to (4) as the *maximum* $d_{\min}$ *precoder* ($\max - d_{\min}$).

Unfortunately, finding $\mathbf{F}_{\max - d_{\min}}$ is known to be NP-hard [6]. The exact expression of $\mathbf{F}_{\max - d_{\min}}$ is only available for $N_{\mathrm{T}} = 2$ and $M = 2, 4, 16$ [7,8]. In the next subsection, we will review a suboptimal precoder which is valid for $N_{\mathrm{T}} > 2$.

C. Equal-Minimum Distance Precoding ($\mathrm{E} - d_{\min}$)

We consider even number of data streams, $N_{\mathrm{T}} \geq 4$, for large MIMO systems. Instead of precoding the full N_{T} data streams, a promising practical approach is to independently encode every two data streams [9–11]. In this way, the complicated problem of finding a $N_{\mathrm{T}} \times N_{\mathrm{T}}$ precoding matrix is reduced to the simple one of finding a 2×2 precoding matrix.

In details, let $\mathbf{H} = \mathbf{U \Sigma V}^H$ be SVD of $\mathbf{H}$, where $\mathbf{U} \in \mathbb{C}^{N_{\mathrm{R}} \times N_{\mathrm{R}}}$, $\mathbf{V} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}$ are unitary matrices, and $\mathbf{\Sigma}$ is an $N_{\mathrm{R}} \times N_{\mathrm{T}}$ rectangular diagonal matrix with non-negative singular values $\{\sigma_i\}_{i=1}^{N_{\mathrm{T}}}$ on the diagonal, with $\sigma_1 \geq \sigma_2 \cdots \geq \sigma_{N_{\mathrm{T}}} \geq 0$. We can also decompose the precoding matrix as $\mathbf{F} = \mathbf{VP}$, where $\mathbf{P} \in \mathbb{C}^{N_{\mathrm{T}} \times N_{\mathrm{T}}}$ is the effective precoding matrix. Then, the equivalent system model is then given by

$$\mathbf{r} = \mathbf{U}^H \mathbf{y} = \mathbf{\Sigma P x} + \mathbf{w}, \quad (7)$$

where $\mathbf{w} \in \mathbb{C}^{N_{\mathrm{T}} \times 1}$ is the equivalent AWGN vector, having the same statistics as $\mathbf{n}$.

The effective precoding matrix $\mathbf{P}$ is designed to form $N_{\mathrm{T}}/2$ pairs of singular values resulting in $N_{\mathrm{T}}/2$ parallel 2×2 MIMO subsystems. We define a pairing $\mathcal{L}$ as a list of $N_{\mathrm{T}}/2$ pairs:

$$\mathcal{L} = \left\{ (\sigma_{i_1}, \sigma_{j_1}), (\sigma_{i_2}, \sigma_{j_2}), \ \cdots, \ (\sigma_{i_{N_{\mathrm{T}}/2}}, \sigma_{j_{N_{\mathrm{T}}/2}}) \right\}, \quad (8)$$

where the k^{th} pair in $\mathcal{L}$ satisfies

$$(i_k, j_k) \in [1, N_{\mathrm{T}}/2] \times [1, N_{\mathrm{T}}/2], \ i_k < j_k, \ k = 1, \ \cdots, \ N_{\mathrm{T}}/2$$

Then (7) can be equivalently written as

$$\begin{bmatrix} r_{i_k} \\ r_{j_k} \end{bmatrix} = \sqrt{E_{\mathrm{T},k}} \begin{bmatrix} \sigma_{i_k} & 0 \\ 0 & \sigma_{j_k} \end{bmatrix} \mathbf{P}_k \begin{bmatrix} x_{i_k} \\ x_{j_k} \end{bmatrix} + \begin{bmatrix} w_{i_k} \\ w_{j_k} \end{bmatrix}, \quad (9)$$

where $\mathbf{P}_k$ is the 2×2 precoding matrix for the k^{th} subsystem, with the power constraint $\|\mathbf{P}_k\|_{\mathrm{F}}^2 = 1$. The coefficient $E_{\mathrm{T},k}$ is the power allocated to the k^{th} subsystem, under the power constraint $\sum_{i=1}^{N_{\mathrm{T}}/2} E_{\mathrm{T},k} = E_{\mathrm{T}}$.

Then, the k^{th} subsystem in (9) gives the minimum distance

$$d_{\min,k} = \sqrt{E_{\mathrm{T},k}} \delta_k, \quad (10)$$

where

$$\delta_k \triangleq d_{\min}\left(\begin{bmatrix} \sigma_{i_k} & 0 \\ 0 & \sigma_{j_k} \end{bmatrix} \mathbf{P}_k \right). \quad (11)$$

The minimum distance of the whole system in (7) is

$$d_{\min}(\mathbf{\Sigma P}) = \min_k d_{\min,k} = \min_k \sqrt{E_{\mathrm{T},k}} \delta_k. \quad (12)$$

Clearly, the criterion for power allocation is the maximization of $d_{\min}(\mathbf{\Sigma P})$, which consists in equalizing the distances $\sqrt{E_{\mathrm{T},k}} \delta_k$ [9]. The optimized minimum distance is shown to be [9, Section IV-B]:

$$d_{\min}(\mathbf{\Sigma P}) = \sqrt{E_{\mathrm{T}} \left(\sum_{i=1}^{N_{\mathrm{T}}/2} \frac{1}{\delta_k^2} \right)^{-1}}. \quad (13)$$

Due to the minimum distance equalization process, this design is named as *equal* $d_{\min}$ *precoder* ($\mathrm{E} - d_{\min}$).

D. Open Problems and Motivations

The minimum distance of $\mathrm{E}-d_{\min}$ precoder in (13) is determined by the list of subchannel pairings and the minimum distance of each subsystem. There are two major unsolved problems in the design of $\mathrm{E}-d_{\min}$ precoder:

1) *Subchannel pairing scheme* $\mathcal{L}$: Equal power allocation will maximize the minimum distance of a given subchannel pairing scheme in (8). However, different pairings will result in different $d_{\min}(\mathbf{\Sigma P})$ in (13). Thus, the criterion of subchannel pairing is the maximization of $d_{\min}(\mathbf{\Sigma P})$. This optimization is expressed as
$$\mathcal{L}_{\text{opt}} = \max_{\mathcal{L}} \sqrt{E_{\mathrm{T}} \left(\sum_{i=1}^{N_{\mathrm{T}}/2} \frac{1}{\delta_k^2} \right)^{-1}}. \tag{14}$$
However, determining $\mathcal{L}_{\text{opt}}$ for a given precoder $\mathbf{P}$ remains an open problem. In the literature, a commonly used heuristic pairing is [9–11]
$$\hat{\mathcal{L}} = \left\{ (\sigma_1, \sigma_{N_{\mathrm{T}}}), (\sigma_2, \sigma_{N_{\mathrm{T}}-1}), \ \cdots, \ (\sigma_{N_{\mathrm{T}}/2}, \sigma_{N_{\mathrm{T}}/2+1}) \right\}, \tag{15}$$
i..e, pairing the largest singular value with the smallest.
2) *Subsystem precoder* $\mathbf{P}_k$: The optimal 2-dimensional precoder is only known for BPSK, 4-QAM, and 16-QAM [7,8]. Suboptimal designs for larger QAM constellations are given in [12]. All these precoders are complex-valued. Hence, the ML detection for each 2×2 MIMO subsystem employs a search over 4-dimensional space. The corresponding complexity of an exhaustive search ML is proportional to M^2.

The above open issues motivate the construction of a new $\mathrm{E}-d_{\min}$ based precoder, with an explicit construction, a provably optimal subchannel pairing scheme, low ML-decoding complexity, and valid for any order QAM modulations.

III. XY-Precoder

In this section, we give the explicit construction of the proposed XY-precoder. We first construct a pair of new X- and Y-precoders, and then combine them to build the XY-precoder. For each precoder, we first derive the minimum distance for each subsystem, and then find the optimal pairing scheme that maximizes the minimum distance of the whole system.

Our precoder is significantly different from the one in [9]. Firstly, the paring scheme in [9] is not optimal, since it targets to maximize a lower bound on the minimum distance [9, Eq.(22)]. In contrast, our paring scheme is optimal, since it maximizes the exact minimum distance. Secondly, the precoder in [9] is complex-valued, thus requires a $4-$D real ML decoder with complexity $O(M^2)$. Our precoder, instead, is real-valued. As shown in [10], the ML decoder can be separated into independent ML decoding of the real and imaginary components of the transmitted symbols. Therefore, our precoder only requires a $2-$D real ML decoder with complexity $O(M)$.

A. X-Precoder

For the k^{th} subsystem, the proposed X-precoder is
$$\mathbf{P}_{\mathrm{X},k} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\dfrac{\sigma_{j_k}}{\sigma_{i_k} + \sigma_{j_k}}} & \sqrt{\dfrac{\sigma_{i_k}}{\sigma_{i_k} + \sigma_{j_k}}} \\ -\sqrt{\dfrac{\sigma_{i_k}}{\sigma_{i_k} + \sigma_{j_k}}} & \sqrt{\dfrac{\sigma_{j_k}}{\sigma_{i_k} + \sigma_{j_k}}} \end{bmatrix}, \tag{16}$$
where $\|\mathbf{P}_{\mathrm{X},k}\|_{\mathrm{F}}^2 = 1$ satisfies the power constraint. The corresponding minimum distance is given in the following lemma.

Lemma 1: With X-precoder, the minimum distance of the k^{th} subsystem is
$$d_{\min,\mathrm{X},k} = \sqrt{E_{\mathrm{T},k}}\delta_{\mathrm{X},k}, \text{ where } \delta_{\mathrm{X},k} = \sqrt{\frac{2}{K}\sigma_{i_k}\sigma_{i_k}}, \tag{17}$$
and K is given in (1).

Proof: See Appendix A. ■

Remark 1: A similar precoder has been proposed in [13]. However, the optimality of subchannel pairings has not been studied. Our contribution is to determine the optimal pairing scheme that maximizes the global minimum distance in (14):
$$\mathcal{L}_{\text{opt},\mathrm{X}} = \max_{\mathcal{L}} \min_{k} d_{\min,\mathrm{X},k} = \max_{\mathcal{L}} \sqrt{E_{\mathrm{T}} \left(\sum_{i=1}^{N_{\mathrm{T}}/2} \frac{1}{\delta_{\mathrm{X},k}^2} \right)^{-1}}. \tag{18}$$

Theorem 1: With X-precoder, the solution of (18) is
$$\mathcal{L}_{\text{opt},\mathrm{X}} = \left\{ (\sigma_1, \sigma_{N_{\mathrm{T}}}), (\sigma_2, \sigma_{N_{\mathrm{T}}-1}), \ \cdots, \ (\sigma_{N_{\mathrm{T}}/2}, \sigma_{N_{\mathrm{T}}/2+1}) \right\}. \tag{19}$$
The optimized minimum distance in (18) is
$$d_{\min,\mathrm{X}} = \sqrt{\frac{2E_{\mathrm{T}}}{K} \left(\sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_k^{-1} \sigma_{N_{\mathrm{T}}-k+1}^{-1} \right)^{-1}}. \tag{20}$$

Proof: See Appendix B. ■

B. Y-Precoder

For the k^{th} subsystem, we propose a new Y-precoder as
$$\mathbf{P}_{\mathrm{Y},k} = \begin{bmatrix} a_k & \left(1 - M^{-1/2}\right) a_k \\ b_k & 0 \end{bmatrix}, \tag{21}$$
where a_k and b_k are non-negative and satisfy the power constraint $\|\mathbf{P}_{\mathrm{Y},k}\|_{\mathrm{F}}^2 = 1$. We then give the optimal choice of (a_k, b_k) which maximizes the minimum distance.

Lemma 2: With Y-precoder, the minimum distance of the k^{th} subsystem is
$$d_{\min,\mathrm{Y},k} = \sqrt{E_{\mathrm{T},k}}\delta_{\mathrm{Y},k}, \tag{22}$$
where
$$\delta_{\mathrm{Y},k} = \begin{cases} \dfrac{2\sigma_{i_k}}{\sqrt{KT}}, & \kappa > T \\ \dfrac{2\sigma_{i_k}\left(\sqrt{M} - 1\right)}{\sqrt{KT + K\kappa\left(M - 2\sqrt{M}\right)}}, & \kappa \leq T \end{cases} \tag{23}$$
$$T = M + \left(\sqrt{M} - 1\right)^2, \tag{24}$$
and $\kappa = \sigma_{i_k}^2 / \sigma_{j_k}^2$. The optimal choice of (a_k, b_k) is
$$\begin{cases} \left(\sqrt{M/T}, 0\right), & \kappa > T \\ \left(\left(T/M + \kappa\left(1 - 2/\sqrt{M}\right)\right)^{-1/2}, \sqrt{1 - a_k^2 T/M}\right), & \kappa \leq T \end{cases}$$

Proof: Due to space limitation, we only give the sketch of the proof. We first derive the expression for $d^2_{\min,\mathrm{Y},k}$:

$$d^2_{\min,\mathrm{Y},k} = \frac{4E_{\mathrm{T},k}}{K} \min\left(\frac{1}{M}\sigma^2_{i_k}a^2_k + \sigma^2_{j_k}b^2_k, \left(1-\frac{1}{\sqrt{M}}\right)^2\sigma^2_{i_k}a^2_k\right).$$

Under the power constraint $a_k^2(1+(1-1/\sqrt{M})^2)+b_k^2 = 1$, we then find the optimal a_k and b_k that maximize $d_{\min,\mathrm{Y},k}$. ■

We then determine the optimal pairing scheme that maximizes the global minimum distance in (14):

$$\mathcal{L}_{\mathrm{opt,Y}} = \max_{\mathcal{L}}\min_{k} d_{\min,\mathrm{Y},k} = \max_{\mathcal{L}}\sqrt{E_{\mathrm{T}}\left(\sum_{k=1}^{N_{\mathrm{T}}/2}\frac{1}{\delta^2_{\mathrm{Y},k}}\right)^{-1}}, \tag{25}$$

Theorem 2: With Y-precoder, when $M=4$, the solution of (25) is

$$\mathcal{L}_{\mathrm{opt,Y}} = \left\{(\sigma_1,\sigma_{\mathcal{P}(1)}),(\sigma_2,\sigma_{\mathcal{P}(2)}),\ \cdots,\ (\sigma_{N_{\mathrm{T}}/2},\sigma_{\mathcal{P}(N_{\mathrm{T}}/2)})\right\}. \tag{26}$$

where the sequence $\left\{\sigma_{\mathcal{P}(1)},...,\sigma_{\mathcal{P}(N_{\mathrm{T}}/2)}\right\}$ represents an arbitrary permutation of $\left\{\sigma_{N_{\mathrm{T}}/2+1},...,\sigma_{N_{\mathrm{T}}}\right\}$. The optimized minimum distance in (25) is

$$d_{\min,\mathrm{Y}} = \sqrt{\frac{4E_{\mathrm{T}}}{KT}\left(\sum_{k=1}^{N_{\mathrm{T}}/2}\frac{1}{\sigma^2_k}\right)^{-1}}, \tag{27}$$

where T is given in (24).

Proof: See Appendix C. ■

Remark 2: The optimal pairing scheme of Y-precoder is not unique. We observe that the optimal pairing for X-precoding $\mathcal{L}_{\mathrm{opt,X}}$ in Theorem 1 is also optimal for Y-precoder.

We then study the minimum distance (25) with $M \geq 4$ and the pairing scheme given in Theorem 2.

Theorem 3: With Y-precoder and the near-optimal pairing scheme in (26), for any M, the minimum distance in (13) can be lower- and upper-bounded as:

$$2\sigma_{\frac{N_{\mathrm{T}}}{2}}\sqrt{\frac{2E_{\mathrm{T}}}{N_{\mathrm{T}}KT}} \leq d_{\min,\mathrm{Y}} \leq \frac{2\sigma_{\frac{N_{\mathrm{T}}}{2}}\left(\sqrt{M}-1\right)\sqrt{E_{\mathrm{T}}}}{\sqrt{KT+K\left(M-2\sqrt{M}\right)}}. \tag{28}$$

where T is given in (24).

Proof: Proof is omitted due to space limitation. ■

Remark 3: The bound in (28) guarantees that Y-precoder has a diversity order equal to

$$\left(N_{\mathrm{T}}/2+1\right)\left(N_{\mathrm{R}}-N_{\mathrm{T}}/2+1\right), \tag{29}$$

which is the same as that in [8,9]. It means that for $M>4$, the pairing scheme in Theorem 2 is also near-optimal.

C. XY-Precoder

As shown in Theorems 1 and 2, X- and Y-precoder are optimized by the pairing scheme in (19). To further increase the minimum distance, we can adaptively switch X- and Y-precoder for each 2×2 MIMO subsystem.

The proposed XY-precoder is given by

$$\mathbf{P}_{\mathrm{XY},k} = \begin{cases}\mathbf{P}_{\mathrm{X},k}, & \delta_{\mathrm{X},k}\geq\delta_{\mathrm{Y},k}\\ \mathbf{P}_{\mathrm{Y},k} & \delta_{\mathrm{X},k}<\delta_{\mathrm{Y},k}\end{cases} \tag{30}$$

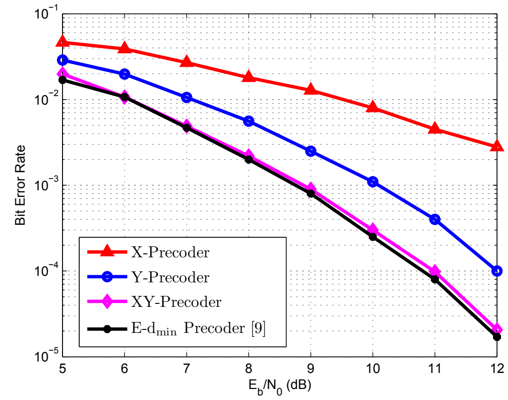

Fig. 1. BER vs. SNR per bit for the uncoded 4×4 system using 4-QAM.

The minimum distance of the k^{th} subsystem is

$$d_{\min,\mathrm{XY},k} = \sqrt{E_{\mathrm{T},k}}\delta_{\mathrm{XY},k}, \text{ where } \delta_{\mathrm{XY},k} = \max\left\{\delta_{\mathrm{X},k},\delta_{\mathrm{Y},k}\right\} \tag{31}$$

The minimum distance in (13) is

$$d_{\min,\mathrm{XY}} = \sqrt{E_{\mathrm{T}}\left(\sum_{k=1}^{N_{\mathrm{T}}/2}\frac{1}{\delta^2_{\mathrm{XY},k}}\right)^{-1}} \geq \max\left\{d_{\min,\mathrm{X}},d_{\min,\mathrm{Y}}\right\}. \tag{32}$$

Example 1: We demonstrate the case $d_{\min,\mathrm{XY}} > d_{\min,\mathrm{Y}}$. When $\sigma_k = \sigma_{N_{\mathrm{T}}-k+1}$ and $M=4$, we have

$$\delta_{\mathrm{XY},k} = \sigma_k\sqrt{\frac{2}{K}} \text{ and } \delta_{\mathrm{Y},k} = \sigma_k\sqrt{\frac{4}{5K}}. \tag{33}$$

We have $\delta_{\mathrm{XY},k} > \delta_{\mathrm{Y},k}$. It is enough to show $d_{\min,\mathrm{XY}} > d_{\min,\mathrm{Y}}$.

IV. Simulation Results

This section examines the performance of XY-precoder, given an exhaustive search ML decoder at the receiver side. For comparison purposes, the performance of the proposed X- and Y-precoder, the best known $\mathrm{E}-d_{\min}$ precoders in [8,9] are also shown. Monte Carlo simulation was used to estimate the bit error rate with Gray mapping.

Fig. 1 and Fig. 2 show the bit error rate for an uncoded system with $N_T = N_R = 4$, $E_{\mathrm{T}} = 4$, using 4-QAM and 16-QAM. Observe that XY-precoder achieves almost the same performance as the best known $\mathrm{E}-d_{\min}$ precoders. The BER curves of Y-precoder has the same slope as the best known ones, thus confirm having the same diversity order. Although X-precoder alone has a weak performance, once combined with Y-precoder, it closes the SNR gap to the best known ones. Note that the proposed X-, Y-, and XY-precoders' exhaustive search ML-decoding complexity is proportional to M, while that of the best known precoders in [8,9] is M^2 $(M=4,16)$.

V. Conclusions

In this work, we have proposed a real-valued precoder for MIMO systems with arbitrary QAM constellations. The advantage of the proposed precoder over the best known $\mathrm{E}-d_{\min}$ precoders is that it has a provably large minimum distance, and it also has lower ML-decoding complexity with same diversity order and a negligible SNR gap. The novelty of our design is to jointly construct a pair of precoders and

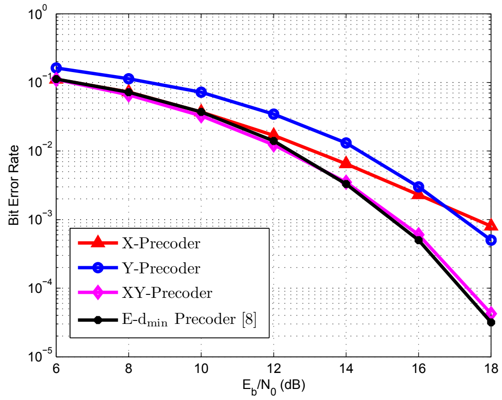

Fig. 2. BER vs. SNR per bit for the uncoded 4×4 system using 16-QAM.

adaptively select the one with larger minimum distance. In particular, our precoders have been proved to share the same optimization strategies, i.e., power allocation and subchannel pairings. This new design criterion guarantees that switching between two precoders is always better than using only one.

Appendix

A. Proof of Lemma 1

Let

$$\mathbf{M}_k = \begin{bmatrix} \sigma_{i_k} & 0 \\ 0 & \sigma_{j_k} \end{bmatrix} \mathbf{P}_{\mathrm{X},k}. \tag{34}$$

Perform QR decomposition, $\mathbf{M}_k = \mathbf{QR}$, where $\mathbf{Q} \in \mathbb{C}^{2\times 2}$ is a unitary matrix and $\mathbf{R} \in \mathbb{C}^{2\times 2}$ is an upper triangular matrix:

$$\mathbf{R} = \frac{1}{\sqrt{2}} \begin{bmatrix} \sqrt{\sigma_{i_k}\sigma_{j_k}} & \sigma_{i_k} - \sigma_{j_k} \\ 0 & \sqrt{\sigma_{i_k}\sigma_{j_k}} \end{bmatrix}. \tag{35}$$

Note that

$$d_{\min}(\mathbf{M}_k) = \min_{\mathbf{s}\in\mathcal{S}^2} \|\mathbf{M}_k \mathbf{s}\| = \min_{\mathbf{s}\in\mathcal{S}^2} \|\mathbf{R}\mathbf{s}\|, \tag{36}$$

where the set $\mathcal{S}$ is given in (6). Since the diagonal elements in $\mathbf{R}$ are equal, according to [14, Property 1], we have

$$d_{\min}(\mathbf{M}_k) = \sqrt{\frac{2}{K}\sigma_{i_k}\sigma_{i_k}} \triangleq \delta_{\mathrm{X},k}. \tag{37}$$

By substituting (37) into (10), we obtain (17). ■

B. Proof of Theorem 1

We can simplify the optimization problem in (18) to

$$\mathcal{L}_{\text{opt,X}} = \min_{\mathcal{L}} \sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_{i_k}^{-1}\sigma_{j_k}^{-1}. \tag{38}$$

We derive the relation

$$\sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_{i_k}^{-1}\sigma_{j_k}^{-1} = \frac{1}{2}\sum_{k=1}^{N_{\mathrm{T}}/2}\left(\sigma_{i_k}^{-1}\sigma_{j_k}^{-1} + \sigma_{j_k}^{-1}\sigma_{i_k}^{-1}\right) = \frac{1}{2}\sum_{t=1}^{N_{\mathrm{T}}} \sigma_t^{-1}\sigma_{\mathcal{P}(t)}^{-1}, \tag{39}$$

where the sequence $\left\{\sigma_{\mathcal{P}(1)}^{-1}, ..., \sigma_{\mathcal{P}(N_{\mathrm{T}})}^{-1}\right\}$ represents a permutation of $\left\{\sigma_1^{-1}, ..., \sigma_{N_{\mathrm{T}}}^{-1}\right\}$, corresponding to $\mathcal{L}$.

Recalling that $\sigma_1^{-1} \le \sigma_2^{-1} \cdots \le \sigma_{N_{\mathrm{T}}}^{-1}$. According to the *rearrangement inequality*, for any permutation $\mathcal{P}$, it holds

$$\frac{1}{2}\sum_{t=1}^{N_{\mathrm{T}}} \sigma_t^{-1}\sigma_{\mathcal{P}(t)}^{-1} \ge \frac{1}{2}\sum_{t=1}^{N_{\mathrm{T}}} \sigma_t^{-1}\sigma_{N_{\mathrm{T}}-t+1}^{-1} = \sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_k^{-1}\sigma_{N_{\mathrm{T}}-k+1}^{-1}. \tag{40}$$

From (39) and (40), for any $\mathcal{L}$, we have

$$\sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_{i_k}^{-1}\sigma_{j_k}^{-1} \ge \sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_k^{-1}\sigma_{N_{\mathrm{T}}-k+1}^{-1}. \tag{41}$$

Therefore, we can obtain (19). ■

C. Proof of Theorem 2

When $M = 4$, using Lemma 2, we have

$$\delta_{\mathrm{Y},k} = \frac{2\sigma_{i_k}}{\sqrt{KT}}, \tag{42}$$

Then, the optimization problem in (25) can be simplified to

$$\mathcal{L}_{\text{opt,Y}} = \min_{\mathcal{L}} \sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_{i_k}^{-2}. \tag{43}$$

Recalling that $\sigma_1^{-2} \le \sigma_2^{-2} \cdots \le \sigma_{N_{\mathrm{T}}}^{-2}$. For any $N_{\mathrm{T}}/2$ non-repeating singular values, we always have

$$\sum_{k=1}^{N_{\mathrm{T}}/2} \sigma_{i_k}^{-2} \ge \sum_{i=1}^{N_{\mathrm{T}}/2} \sigma_i^{-2}. \tag{44}$$

Therefore, we can obtain (26) and (27). ■

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