Algebraic Multiuser Space–Time Block Codes for 2×2 MIMO

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***Abstract*—In this paper, we consider multiuser space-time block codes (STBCs) for 2×2 multiple–input multiple–output (MIMO) uplink transmissions. Using a truncated union–bound (UB) approximation, we propose design criteria of multiuser STBCs for quasi-static fading MIMO multiple access channels (MACs). Next, we demonstrate how, by combining the structure of algebraic perfect STBCs in [10], a family of multiuser STBCs can be constructed to fulfill the design criteria, and show that the proposed STBC outperforms all previously known codes over quasi-static fading MIMO MACs.**

***Index Terms*—space-time block codes, multiuser, MIMO, MAC.**

I. Introduction

Space-time block codes (STBCs) have been intensively studied for *single-user* multiple–input multiple–output (MIMO) [1–10]. Recently, Gärtner and Bölcskei [11] extended the idea of single user STBC to *multiuser* cases. Using a concept of dominant error regions, the design criteria of multiuser STBCs were proposed in [11] to increase information rate over quasi-static fading MIMO multiple access channels (MACs). A 2×2 MIMO multiuser STBC was proposed based on Alamouti structure.

Motivated by the above design criteria, an algebraic construction of multiuser STBCs was presented in [15] to achieve the diversity-multiplexing tradeoff for users using a single transmit antenna ($n_t = 1$) and any number of receive antennas n_r.

Another family of multiuser STBCs were proposed in [13], where the design criteria were generalized for more than two users from the perspective of minimizing an upper bound of pairwise error probability (PEP). However, these codes incur in large *peak-to-average penalties*, since some elements in the codeword matrices are zero.

In our paper, we consider 2×2 multiuser MIMO over quasi-static fading MACs, which was also discussed in [11, 13]. Unlike the multiuser codes in [11, 13], we propose the code design criteria based on a truncated union-bound (UB) approximation. Motivated by algebraic perfect space–time block codes in [10], we demonstrate the construction of a family of multiuser STBCs in order to minimize the error probability of the truncated UB, without the peak-to-average penalty of [13]. Within this family, we present a code design example for a two–user 2×2 MIMO case. Finally we show by simulation that the proposed codes outperform the previously known STBCs [11, 13].

The outline of this paper is organized as follows. Section II introduces system model. In Section III we present the design criteria of algebraic multiuser MIMO STBCs. In Section IV, we show a design example of two-user 2×2 STBCs, which outperform previously known multiuser MIMO STBCs in [11, 13]. Finally, conclusions are drawn in Section V.

Notations: Boldface letters are used for column vectors, and capital boldface letters for matrices. Superscripts ^{T} and † denote transposition and Hermitian transposition, respectively. Let $\mathbb{Q}$ and $\mathbb{C}$ denote the field of rational and complex numbers, respectively. The $\mathrm{vec}(\cdot)$ operator stacks the m column vectors of a $n\times m$ complex matrix into a mn complex column vector. Let $\|\cdot\|$ denote the Frobenius norm and let $\mathbb{E}[\cdot]$ denote mean of a random variable.

II. System Model

We consider an uplink scenario, where K uncoordinated users simultaneously communicate with a base station over a quasi-static fading MIMO MAC. We assume that each user employs an identical $n_r \times n_t$ MIMO system.

A. Transmitter

At the transmitter of the k-th user, let

$$\mathbf{s}^{(k)} \triangleq \left[s_1^{(k)}, \ldots, s_i^{(k)}, \ldots, s_N^{(k)}\right]^T \in \mathbb{C}^N$$

be the information symbol vector of length N, where $s_i^{(k)}$, $i = 1, \ldots, N$, denote independent information symbols drawn from a complex $Q-$QAM constellation.

Then, for any k-th user, the symbol vector $\mathbf{s}^{(k)}$ is encoded by its individual STBC. This produces the k-th *user codeword matrix* $\mathbf{C}_k \in \mathbb{C}^{n_t\times N}$ from the codebook $\mathcal{C}_k$ spanning over N channel uses, defined as

$$\mathbf{C}_k \triangleq \left[\left(\mathbf{c}_1^{(k)}\right)^T, \ldots, \left(\mathbf{c}_j^{(k)}\right)^T, \ldots, \left(\mathbf{c}_{n_t}^{(k)}\right)^T\right]^T \in \mathcal{C}_k$$

where $\mathbf{c}_j^{(k)} \triangleq \{c_{j,n}^{(k)}\}^T \in \mathbb{C}^N, j = 1, \ldots, n_t$, and $c_{j,n}^{(k)}$ denotes the space-time block coded symbol transmitted at the j-th transmit antenna of user k over the n-th channel use. The above choice implies that the users transmit at a rate of one symbol per channel use.

All K users are assumed to simultaneously transmit their codeword matrices $\mathbf{C}_k$ yielding the following *joint codeword matrix*

$$\mathbf{X} \triangleq [\mathbf{C}_1^T, \ldots, \mathbf{C}_k^T, \ldots, \mathbf{C}_K^T]^T \in \mathcal{C} \quad (1)$$

where $\mathcal{C}$ is the *joint codebook*. In this paper, we assume that each user employs a linear STBC [12, Definition 5], so that the elements $c_{j,n}^{(k)}$ are linear combinations of N complex Q–QAM symbols.

B. Receiver

At the receiver, the received signal matrix $\mathbf{Y} \in \mathbb{C}^{n_r \times N}$ can be written as

$$\mathbf{Y} \;=\; \mathbf{H}\mathbf{X} + \mathbf{N}, \quad (2)$$

where $\mathbf{N} \in \mathbb{C}^{n_r \times N}$ is the complex white Gaussian noise with i.i.d. entries $\sim \mathcal{N}_{\mathbb{C}}(0, N_0)$ and $\mathbf{H} \in \mathbb{C}^{n_r \times K n_t}$ is defined as

$$\mathbf{H} = \left[\mathbf{H}^{(1)}, \ldots, \mathbf{H}^{(k)}, \ldots, \mathbf{H}^{(K)}\right]$$

where $\mathbf{H}^{(k)} \triangleq \{H_{i,j}^{(k)}\} \in \mathbb{C}^{n_r \times n_t}$, denotes the channel matrix associated with the k-th user, assumed to remain constant during the transmission of a codeword of user k, and to take on independent values from one codeword to another of user k. The elements $H_{i,j}^{(k)}$ are the channel coefficients from the j-th transmit antenna to the i-th receive antenna for user k, assumed to be i.i.d. circularly symmetric Gaussian random variable $\sim \mathcal{N}_{\mathbb{C}}(0, 1)$. The channel matrices of all K users are assumed to be perfectly known at the receiver, but not at the transmitter.

III. New Multiuser Space-Time Block Codes

In this Section, we present the jointly full-rank design and code design criteria, respectively.

Let us consider all K users, assuming that a joint codeword matrix $\mathbf{X} \in \mathcal{C}$ is transmitted, it may occur that

$$\|\mathbf{Y} - \mathbf{H}\mathbf{X}\|^2 > \|\mathbf{Y} - \mathbf{H}\widehat{\mathbf{X}}\|^2$$

with $\mathbf{X} \neq \widehat{\mathbf{X}}$, resulting in a *pairwise error*.

Let $\mathbf{X} - \widehat{\mathbf{X}}$, with $\mathbf{X} \neq \widehat{\mathbf{X}}$, be the *joint codeword-difference matrix* and let

$$\mathbf{A} \triangleq (\mathbf{X} - \widehat{\mathbf{X}})(\mathbf{X} - \widehat{\mathbf{X}})^\dagger$$

be the *joint codeword-distance matrix*.

Similarly, when only user k is in error, assuming that a codeword matrix $\mathbf{C}_k \in \mathcal{C}_k$ is transmitted and $\widehat{\mathbf{C}}_k$ is erroneously detected at the receiver, we call $\mathbf{C}_k - \widehat{\mathbf{C}}_k$ the *user codeword difference matrix*. The corresponding *user codeword distance matrix* is defined as

$$\mathbf{E}^{(k)} \triangleq (\mathbf{C}_k - \widehat{\mathbf{C}}_k)(\mathbf{C}_k - \widehat{\mathbf{C}}_k)^\dagger$$

Let r_k denote the minimum rank of $\mathbf{E}^{(k)}$ for all user codeword pairs in $\mathcal{C}_k$. We will assume $r_k = \min(n_t, N) = r$ for all k, i.e., all user codes have *full–rank*.

If this full–rank condition holds for all K users, it is not guaranteed that the joint code is also full rank. We say a multiuser STBC is *jointly full–rank*, if all $\mathbf{E}^{(k)} \neq \mathbf{0}$ then $\text{rank}(\mathbf{A}) = Kr$. Note that this property still holds for any subset of the K users. We will show in the following how to design the such codes.

A. Jointly full-rank design

Here, we assume that there are K users each with $n_t = 2$ antennas, a receiver with $n_r = 2$ antennas. We choose $N = 2K$ channel uses so that the joint codeword matrix $\mathbf{X}$ is a square matrix. Given the k-th user information symbol vector $\mathbf{s}^{(k)}$, we use an algebraic unitary matrix $\mathbf{M}$ with *full diversity* (see reference in [14]) to generate

$$\mathbf{v}^{(k)} = \mathbf{M}\mathbf{s}^{(k)} = \left[v_1^{(k)}, \ldots, v_N^{(k)}\right]^T \qquad k = 1, \ldots, K \quad (3)$$

The matrix $\mathbf{M}$ is obtained from the canonical embedding of an integral basis $\{\omega_j\}$, $j = 1, \ldots, N$ of an ideal of an algebraic number field L of degree N over $\mathbb{Q}(i)$ [16]. The full diversity property implies that all the elements of $\mathbf{v}^{(k)}$ are non-zero for any non-zero information vector $\mathbf{s}^{(k)}$ [16]. The user codewords are generated as

$$\mathbf{C}_1 = \begin{bmatrix} v_1^{(1)} & v_2^{(1)} & \cdots & & v_N^{(1)} \\ \gamma v_N^{(1)} & v_1^{(1)} & & \cdots & v_{N-1}^{(1)} \end{bmatrix}$$

$$\mathbf{C}_2 = \begin{bmatrix} \gamma v_{N-1}^{(2)} & \gamma v_N^{(2)} & \cdots & & v_{N-2}^{(2)} \\ \gamma v_{N-2}^{(2)} & \gamma v_{N-1}^{(2)} & \gamma v_N^{(2)} & \cdots & v_{N-3}^{(k)} \end{bmatrix}$$

$$\mathbf{C}_3 = \cdots$$

where $\gamma \neq 1$ is a complex number on the unit circle in order to preserve a uniform transmitted power from each antenna. In such a manner, the code will not incur in extra peak–to–average penalty, since all entries are non-zero with the same average power (see design example for details).

Lemma 1: For $\gamma \neq 1$, the above user codes $\mathbf{C}_k$ are full rank $r = 2$ for all K users. ■

Proof. It is enough to show that the two rows of $\mathbf{C}_k$ are linearly independent, which is equivalent to saying they can not be scalar multiples for any non zero information vector $\mathbf{s}^{(k)}$. This is the case thanks to the term $\gamma \neq 1$ which multiplies a different number of elements in each row. ■

Lemma 2: If $N = n_t K$ the joint codeword matrices $\mathbf{X}$ are square and the multiuser code $\mathcal{C}$ is jointly full–rank if γ is transcendental. ■

Proof. Looking at the structure of the $N \times N$ square codeword matrix $\mathbf{X}$ we note that the elements of the lower triangular part are multiplied by γ. It can be easily verified that the determinant of $\mathbf{X}$ is a polynomial $p(\gamma)$ in the variable γ by using the well known expression

$$\det(\mathbf{X}) = \sum_{\pi} \prod_{i=1}^{N} x_{i,\pi(i)}$$

where the sum runs over all the permutations π. This polynomial has degree $N-1$ since the coefficient of the term γ^{N-1} is given by

$$x_{1,N} \cdot x_{2,1} \cdot x_{3,2} \cdots x_{N,N-1} \neq 0$$

which is not zero thanks to the full diversity rotation in (3), yielding vectors $\mathbf{v}^{(k)}$ with all non-zero entries. The coefficients of $p(\gamma)$ are in the algebraic number field L defined after equation (3). The roots of the polynomial equation $p(\gamma) = 0$ are in some algebraic extension L' of L [16]. By choosing γ to be transcendental (i.e. in no finite extension of L) we can guarantee that

$$p(\gamma) = \det(\mathbf{X}) \neq 0$$

■

Note that the above Lemma gives only a necessary condition and some specific not transcendental γs not belonging to L' can also yield a jointly full–rank multiuser code.

B. Design criteria

To simplify analysis, we assume that the jointly full–rank multiuser STBC is *linear* [12]. Then, the error probability of the multiuser MIMO is upper bounded by the following union bound [13]:

$$\begin{aligned} P(e) &\leq \sum_{\mathbf{X}\neq\mathbf{0}} P\left(e|\mathbf{X}\right) \\ &\leq \sum_{\mathbf{X}\neq\mathbf{0}} \sum_{k=1}^{K} \sum_{(i_1,\ldots,i_k)}^{A_k} P\left(e_{i_1} \cap \cdots \cap e_{i_k}|\mathbf{X}\right) \end{aligned} \quad (4)$$

where e_k represents the k-th user error event, the sum $\sum_{(i_1,\ldots,i_k)}^{A_k}$ is over all $A_k \triangleq \binom{K}{k}$ possible k-tuples of users in error. The k-tuple $(i_1, \ldots, i_k)$ denotes the indices of k distinct users. Using the Chernoff bound, we can upper bound each term in (4) with:

$$P\left(e_{i_1} \cap \cdots \cap e_{i_k}|\mathbf{X}\right) \leq \left(\frac{E_s}{N_0}\right)^{-n_r k r} \left[\delta_{(i_1,\ldots,i_k)}(\mathbf{X})\right]^{-n_r} \quad (5)$$

where

$$E_s \triangleq \frac{1}{K n_t N} \sum_{i,j} \mathbb{E}[|x_{i,j}|^2]$$

is the average energy per QAM information symbol and the determinants:

$$\delta_{(i_1,\ldots,i_k)}(\mathbf{X}) \triangleq \det\left(\sum_{\ell=1}^{k} \mathbf{C}_{i_\ell}\mathbf{C}_{i_\ell}^{\dagger}\right) \quad (6)$$

We can further define the corresponding *minimum determinants* among all the k-tuples

$$\delta_k^{(\min)} = \min_{\substack{(i_1,\ldots,i_k) \\ \mathbf{X}\neq\mathbf{0}}} \delta_{(i_1,\ldots,i_k)}(\mathbf{X})$$

Finally, we consider a truncated union bound based only on the terms corresponding to the minimum determinants $\delta_k^{(\min)}$

$$P(e) \approx \sum_{k=1}^{K} A_k B_k P(\delta_k^{(\min)})$$

where the $A_k B_k$ is the multiplicity of the term

$$P(\delta_k^{(\min)}) = \left(\frac{E_s}{N_0}\right)^{-n_r k r} \left(\delta_k^{(\min)}\right)^{-n_r} \quad (7)$$

which represents the dominant error probability of a k-tuple of users.

The codes designed in the previous section satisfy the following lemma.

Lemma 3: The determinants in (6) are all non-zero. ■

Proof. Since the terms $\mathbf{C}_{i_\ell}\mathbf{C}_{i_\ell}^{\dagger}$ in (6) are positive definite we use the determinant inequality

$$\det\left(\sum_{\ell=1}^{k} \mathbf{C}_{i_\ell}\mathbf{C}_{i_\ell}^{\dagger}\right) \geq \sum_{\ell=1}^{k} \det\left(\mathbf{C}_{i_\ell}\mathbf{C}_{i_\ell}^{\dagger}\right)$$

where the determinants on the rhs are all greater than zero due to Lemma 1. ■

Hence, under the full–rank and linearity assumption, in order to minimize the error probability $P(e)$, we should design multiuser STBCs to

1) maximize the minimum determinants $\delta_k^{(\min)}$, $\forall k$;
2) minimize the associated multiplicity $A_k B_k$.

IV. Code Design Example

As an example, we consider $K = 2$ users each employing a 2×2 MIMO with $N = 4$ over quasi-static fading MACs. The unitary matrix $\mathbf{M}$ in [10] is chosen and $\gamma = i$. Note that this γ is not transcendental but still guarantees the non-zero determinant. We also note that the proposed code and the known codes in [11, 13] are "full–rank" joint multiuser STBCs, i.e., $r_k = 2$ and $\text{rank}(\mathbf{A}) = 4$. We recall that the error probability $P(e)$ takes into account the total number of errors of both users.

Let us define the *peak-signal-to-noise ratio* as

$$\text{Peak-SNR} \triangleq n_t E_p / N_0$$

where

$$E_p = \max_{i,j} \mathbb{E}[|x_{i,j}|^2]$$

denotes the *peak average energy* of a transmitted QAM symbol from one antenna. We have

$$E_p = E_s$$

for the proposed code and the one in [11], while

$$E_p = 2E_s$$

for the code in [13] which has some zero entries in the codeword.

In Table I, the proposed STBCs together with multiuser STBCs in [11, 13] are compared in terms of

1) the minimum determinant $\delta_k^{(\min)}$ when k users are simultaneously in error;
2) the associated multiplicities $A_k B_k$ and the SNR (dB) at codeword error rate (CER) of 10^{-3}.

We see in Table I that

1) when one user is in error, the minimum determinant of the code of [11] is slightly larger than that of the proposed code;

Codes	$\delta_1^{(\min)}$	A_1B_1	$\delta_2^{(\min)}$	A_2B_2
New	13.2	16	52	16
GB	16	64	32	256
ZL	4	64	8	256

TABLE I
COMPARISON OF MINIMUM DETERMINANTS WHEN ONE OR TWO USERS ARE IN ERROR, ASSOCIATED MULTIPLICITIES.

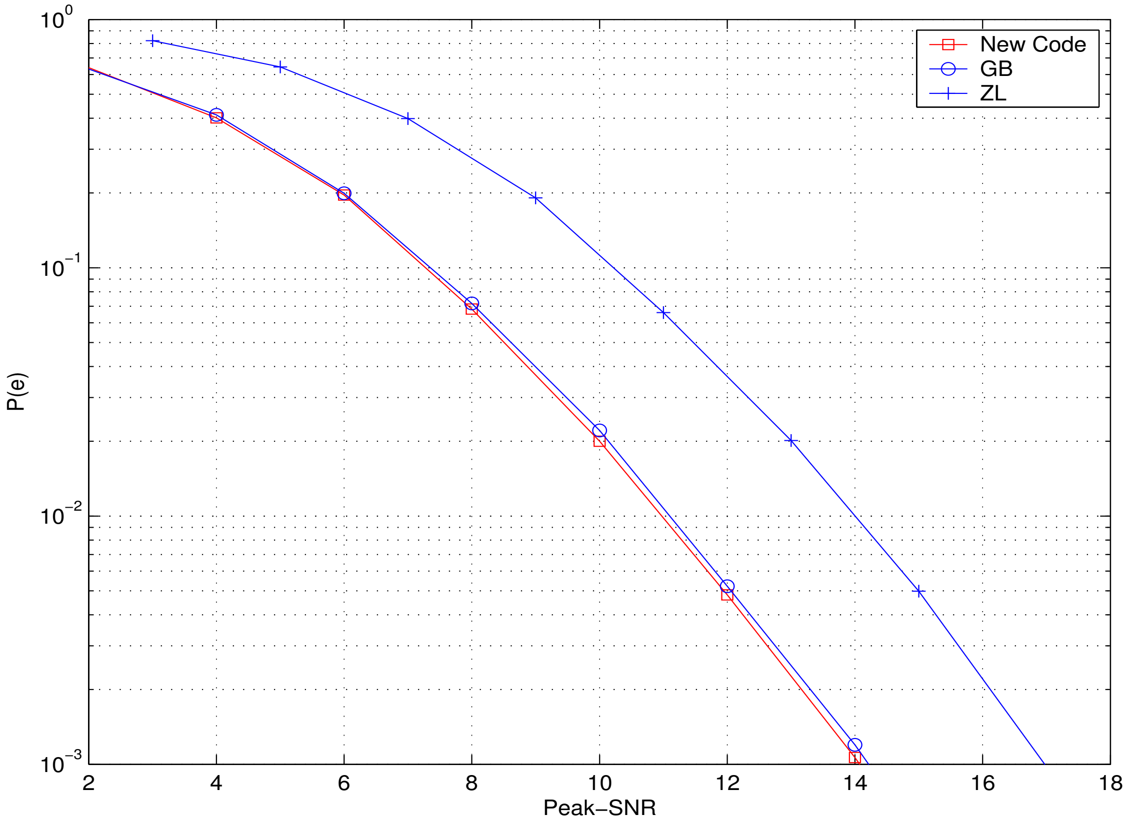

Fig. 1. Comparison of the CER performance of the new code, known codes in [11] and [13], 4-QAM signalling, quasi-static fading channel.

2) when both users are in error, the minimum determinants of our code is the largest among all multiuser STBCs.

In both conditions, the associated multiplicities of the proposed code are significantly smaller than those of [11, 13].

With 4-QAM signalling, we show CER performance of the proposed code and other previously known codes in Fig. 1. At CER$= 10^{-3}$, it is shown that the proposed code outperforms 3dB to that of the code in [13], while it also outperforms slightly to that of the code in [11].

V. CONCLUSION

In this paper, we propose new algebraic multiuser 2×2 STBCs for quasi-static MIMO MACs. Using a UB approximation, we first present the code design criteria. Combining algebraic perfect STBC structures, we show how to design a family of multiuser STBCs to satisfy the design criteria, yet without peak–to–average penalty. Within this family, we present a code design example for a two-user case. It is shown that the proposed multiuser STBC for quasi–static fading outperforms all previously known codes.

ACKNOWLEDGMENT

The authors would like to thank M.E. Gärtner and H. Bölcskei for their fruitful discussions. The work of E. Viterbo was supported by the STREP project No. IST-026905 (MASCOT) within the Sixth Framework Programme of the European Commission.

REFERENCES

[1] S. M. Alamouti, "A simple transmit diversity technique for wireless communications," *IEEE Journal on Selected Areas in Communications*, vol. 16, no. 8, pp. 1451–1458, Oct. 1998.

[2] V. Tarokh, H. Jafarkhani, A. R. Calderbank, “Space–time block codes from orthogonal designs,” *IEEE Transcations on Information Theory*, vol. 45, no. 5, pp. 1456–1467, Jul 1999.

[3] V. Tarokh, N. Seshadri and A. R. Calderbank, “Space–time Codes for High Data Rate Wireless Communications: Performance Criterion and Code Construction,” *IEEE Transactions on Information Theory*, vol. 44, no. 2, pp. 744–765, 1998.

[4] J.-C. Guey, M.P. Fitz, M.R. Bell and W.-Y. Guo, “Signal design for transmitter diversity wireless communication Systems over Ranleith fading channels,” *IEEE Transactions on Communications*, vol. 47, no. 4, pp. 527–537, 1999.

[5] H. Jafarkhani, “A quasi-orthogonal space–time block code,” in *IEEE Communication Letters*, vol. 49, no. 1, pp. 1–4, Jan. 2001.

[6] H. El Gamal and M. O. Damen, “Universal space–time codes,” *IEEE Transactions on Information Theory*, vol. 49, no. 5, pp. 1097–1119, Man 2003.

[7] B. A. Sethuraman, B. S. Rajan, and V. Shashidhar, “Full-diversity, high-rate space–time block codes from division algebras,” *IEEE Transactions on Information Theory*, vol. 49, pp. 2596–2616, Oct. 2003.

[8] J.-C. Belfiore, G. Rekaya, and E. Viterbo, “The Golden Code: A 2 × 2 full-rate space–time code with non-vanishing determinants,” *IEEE Transactions on Information Theory*, vol. 51, no. 4, pp. 1432–1436, Apr. 2005.

[9] P. Elia, K.R. Kumar, S.A. Pawar, P.V. Kumar, Hsiao-feng Lu, “Explicit space–time codes that achieve the diversity-multiplexing gain tradeoff,” *IEEE Transactions on Information Theory*, vol. 52, no. 9, pp. 3869–3884, Sept. 2006.

[10] F. Oggier, G. Rekaya, J.-C. Belfiore, and E. Viterbo, “Perfect space–time block codes,” *IEEE Trans. Inform. Theory*, vol. 52, n. 9, pp. 3885–3902, Sept. 2006.

[11] M.E. Gärtner and H. Bölcskei, “Multiuser space-time/frequency code design,” *Proc. IEEE Int. Symp. on Inform. Theory (ISIT)*, Seattle, WA, pp. 2819-2823, July 2006.

[12] E. Biglieri, Y. Hong, E. Viterbo, “On fast–decodable space–time block codes,” submitted to *IEEE Trans. Inform. Theory*, available in arxiv: CS.IT. 0708.2804.

[13] W. Zhang and K. Ben Letaief, “A systematic design of multiuser space–frequency codes for MIMO–OFDM systems,” in *Proc. IEEE Inte. Conf. on Commun. (ICC’07)*, pp. 1054–1058, July 2007.

[14] J. Boutros and E. Viterbo, “Signal Space Diversity: a power and bandwidth efficient diversity technique for the Rayleigh fading channel,” *IEEE Transactions on Information Theory*, vol. 44, n. 4, pp. 1453–1467, July 1998.

[15] M. Badr and J.-C. Belfiore, “Optimal Space-Time Codes for the non-cooperative MAC channel,” International Zurich Seminar, March 2008.

[16] F. Oggier, E. Viterbo, “Algebraic number theory and code design for Rayleigh fading channels,” in *Foundations and Trends in Communications and Information Theory*, vol. 1, pp. 333–415, 2004.