Asymptotic Analysis of Multidimensional Jittered Sampling

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***Abstract*—We study a class of random matrices that appear in several communication and signal processing applications, and whose asymptotic eigenvalue distribution is closely related to the reconstruction error of an irregularly sampled bandlimited signal. We focus on the case where the random variables characterizing these matrices are d-dimensional vectors, independent, and quasi-equally spaced, i.e., they have an arbitrary distribution and their averages are vertices of a d-dimensional grid. Although a closed form expression of the eigenvalue distribution is still unknown, under these conditions we are able i) to derive the distribution moments as the matrix size grows to infinity, while its aspect ratio is kept constant, and ii) to show that the eigenvalue distribution tends to the Marčenko-Pastur law as $d \rightarrow \infty$. These results can find application in several fields, as an example we show how they can be used for the estimation of the mean square error provided by linear reconstruction techniques.**

***Index Terms*—Error analysis, signal reconstruction, signal sampling.**

I. Introduction

CONSIDER the class of random matrices[1] of size $(2M+1) \times r$, with entries given by

$$(\mathbf{G})_{\ell,q} = \frac{\exp(-\mathrm{j}2\pi\ell x_q)}{\sqrt{2M+1}} \tag{1}$$

for $\ell = -M,\dots,M$, $q = 0,\dots,r-1$. The scalars x_q are independent random variables characterized by a probability density function (pdf) $f_{x_q}(z)$, with $0 \le z \le 1$. These matrices are of Vandermonde type with complex exponential entries; they appear in many signal/image processing applications and have been studied in a number of recent works (see, e.g., [1]–[6]). More specifically, in the field of signal processing for sensor networks, [1] studied the performance of linear reconstruction techniques for physical fields irregularly sampled by sensors. In such scenario, the random variables x_q in (1) represent the coordinates of the sensor nodes. The work in [2] addressed the case where these coordinates are uniformly distributed and subject to an unknown jitter. In the field of communications, the study in [6] presented a number of applications where these matrices appear, which range from multiuser MIMO systems to multifold scattering.

In spite of their numerous applications, few results are known for the Vandermonde matrices in (1). In particular, a closed form expression for the *eigenvalue distribution* of the Hermitian Toeplitz matrix $\mathbf{G}\mathbf{G}^{\mathsf{H}}$, as well as its asymptotic behavior, would be of great interest. As an example, in [1] and [4], it has been observed that the performance of linear techniques for reconstructing a signal from a set of irregularly spaced samples with known coordinates is a function of the *asymptotic eigenvalue distribution* of $\mathbf{G}\mathbf{G}^{\mathsf{H}}$.

In general, given an $N \times N$ Hermitian matrix $\mathbf{X}$, the *empirical cumulative distribution* function (or empirical spectral distribution) of its eigenvalues is defined as[13] $\mathsf{F}_{\lambda}^{N}(z) = 1/N \sum_{i=1}^{N} 1\{\lambda_i(\mathbf{X}) \le z\}$ where $\lambda_1(\mathbf{X}),\dots,\lambda_N(\mathbf{X})$ are the eigenvalues of $\mathbf{X}$ and $1\{\cdot\}$ is the indicator function. Since $\mathbf{X}$ is Hermitian, the function $\mathsf{F}_{\lambda}^{N}(z)$ has support in $z \ge 0$. If $\mathsf{F}_{\lambda}^{N}(z)$ converges as $N \rightarrow \infty$, the corresponding limit is denoted by $\mathsf{F}_{\lambda}(z)$. The asymptotic pdf (i.e., the asymptotic eigenvalue distribution of $\mathbf{X}$) is denoted by $f_{\lambda}(z)$. In particular, for the class of $(2M+1) \times r$ matrices $\mathbf{G}$ defined in (1), the asymptotic eigenvalue distribution of the Hermitian matrix $\mathbf{G}\mathbf{G}^{\mathsf{H}}$ is defined in the limit of M and r growing to infinity while the matrix aspect ratio[2] $(2M+1)/r$ is kept constant.

In this paper, we consider a general formulation which extends the random variables in (1) to a d-dimensional domain: we study the properties of random matrices $\mathbf{G}_d$ of size $(2M+1)^d \times r$ and entries given by

$$(\mathbf{G}_d)_{\nu(\boldsymbol{\ell}),q} = \frac{\exp\left(-\mathrm{j}2\pi\boldsymbol{\ell}^{\mathsf{T}}\mathbf{x}_q\right)}{\sqrt{(2M+1)^d}} \tag{2}$$

where the vectors $\mathbf{x}_q = [x_{q1},\dots,x_{qd}]^{\mathsf{T}}$ have independent entries, characterized by the pdf $f_{x_{qm}}(z)$, $q = 0,\dots,r-1$, $m = 1,\dots,d$, and d is the vector size. The invertible function

$$\nu(\boldsymbol{\ell}) = \sum_{m=1}^{d} (2M+1)^{m-1}\ell_m \tag{3}$$

maps the vector of integers $\boldsymbol{\ell} = [\ell_1,\dots,\ell_d]^{\mathsf{T}}$, $\ell_m = -M,\dots,M$ onto a scalar index, i.e., the row index of the

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[1]Column vectors and matrices are denoted by bold lowercase and bold upper case letters, respectively. $(\mathbf{X})_{kq}$ is the (k,q) entry of the matrix $\mathbf{X}$. The conjugate transpose operator is denoted by $(\cdot)^{\mathsf{H}}$.

[2]The aspect ratio of $\mathbf{G}$ is the ratio between the number of rows and the number of columns of the matrix.

matrix $\mathbf{G}_d$. Notice that, when $d = 1$, $\mathbf{G}_d$ reduces to (1). By defining

$$\beta = \frac{(2M+1)^d}{r}$$

as the aspect ratio of $\mathbf{G}_d$, we consider the properties of the Hermitian random matrix $\mathbf{T}_d = \beta \mathbf{G}_d \mathbf{G}_d^{\mathsf{H}}$ where the coefficient β is used for normalization purposes. In general, the asymptotic eigenvalue distribution of $\mathbf{T}_d$, denoted by $f_\lambda(d, \beta, z)$, depends on the parameters d and β: how to derive an analytic expression of such distribution is still an open problem.

A. Novel Contributions

For the matrix model in (2), we study the interesting case where $\mathbf{x}_q$ are independent, *quasi-equally spaced* random variables in the d-dimensional hypercube $[0,1)^d$. In other words, we assume that the averages of $\mathbf{x}_q$ are the vertices of a d-dimensional grid in $[0,1)^d$. Note that the distribution of the random variables $\mathbf{x}_q$ can be of any kind, the only assumption we make is on their averages being equally spaced. Such kind of matrices appear in many practical applications. For example when analyzing measurement systems affected by jitter, or when considering a sensor network sampling a physical field, where sensors' coordinates are quasi-equally spaced, due to terrain conditions and deployment practicality [7].

Under these conditions on the matrix $\mathbf{G}_d$, the main contributions of this work can be summarized as follows:

- in Section III we derive a closed form expression for the moments of $f_\lambda(d, \beta, z)$;
- this enables us to show in Section IV that, as $d \to \infty$, the asymptotic eigenvalue distribution of $\mathbf{T}_d$ tends to the Marčenko-Pastur law [10];
- moreover in Section V we show some numerical results and present some applications where the moments derived in Section III and the asymptotic approximation found in Section IV can be of great use.

II. Previous Results

Before presenting the details of our novel contributions, we briefly review previous results on the $\mathbf{G}_d$ matrices.

For the case $d = 1$

i) the work in [1] considered an irregularly sampled bandlimited signal, which is reconstructed using linear techniques. The samples coordinates, x_q were assumed to be known. The performance of the reconstruction system was shown to be a function of the eigenvalue distribution $f_\lambda(1, \beta, z)$ of the matrix $\mathbf{T}_1 = \beta \mathbf{G}_1 \mathbf{G}_1^{\mathsf{H}}$;

ii) an explicit expression of the moments $\mathbb{E}[\lambda_{1,\beta}^p] = \int_0^\infty z^p f_\lambda(1, \beta, z) \mathrm{d}z$ was attained in [3], for the specific case where x_q are i.i.d. and uniformly distributed in $[0, 1)$;

iii) in the case where x_q are independent, quasi-equally spaced random variables, the analytic expression of $\mathbb{E}[\lambda_{1,\beta}^2]$, was obtained in [2];

iv) in [5] the moments $f_\lambda(1, \beta, z)$ were derived for i.i.d. random variables x_q with arbitrary distribution $f_{x_q}(z)$.

For the multidimensional case $(d > 1)$,

i) the work in [3] considered the case where the entries of the vectors $\mathbf{x}_q = [x_{q1}, \dots, x_{qd}]^{\mathsf{T}}$ are i.i.d, *uniformly distributed* in the hypercube $[0,1)^d$ and under such assumption derived an analytic expression of the moments of $f_\lambda(d, \beta, z)$, for any given d and β;

ii) in [3] it was also shown that, when the vectors $\mathbf{x}_q$ are i.i.d and uniformly distributed in $[0,1)^d$, $f_\lambda(d, \beta, z)$ tends to the Marčenko-Pastur law [10] as $d \to \infty$.

III. Closed Form Expression of the Moments of the Asymptotic Eigenvalue PDF

Here we first introduce the problem under study and our system assumptions, then we derive an analytic expression of the moments of $f(d, \beta, z)$. In Appendix A we report a list of the main symbols used in the derivation of our results.

A. Problem Formulation

We consider the matrix class in (2) and assume that the r vectors $\mathbf{x}_q$, $q = 0, \dots, r-1$ are independent, quasi-equally spaced random variables in the d-dimensional hypercube $[0,1)^d$, i.e., the averages of $\mathbf{x}$ are the vertices of a d-dimensional grid in $[0,1)^d$.

We define ρ as the number of vertices per dimension, thus, the total number of vertices is $r = \rho^d$. We denote the coordinate of a generic vertex of the grid by the vector $\mathbf{q}/\rho \in [0,1)^d$, where $\mathbf{q} = [q_1, \dots, q_d]^{\mathsf{T}}$, is an integer vector and $q_m = 0, \dots, \rho - 1$. For simplicity and in analogy with (3), we identify the vertex with coordinate $\mathbf{q}/\rho$ by the scalar index

$$\mu(\mathbf{q}) = \sum_{m=1}^{d} \rho^{m-1} q_m. \tag{4}$$

Note that $0 \le \mu(\mathbf{q}) \le r - 1$ is an invertible function that uniquely maps the vector $\mathbf{q}$ to the integer $\mu(\mathbf{q})$, which represents the column index of $\mathbf{G}_d$. Then, we have $\mathbf{x}_{\mu(\mathbf{q})} = \mathbf{q}/\rho + \tilde{\mathbf{x}}_{\mu(\mathbf{q})}/\rho$ where we assume that the entries of the vectors $\tilde{\mathbf{x}}_{\mu(\mathbf{q})}$ are zero mean i.i.d. with pdf $f_{\tilde{x}}(z)$, which does not depend on r, M, or $\mathbf{q}$. The average $\mathbb{E}[\mathbf{x}_{\mu(\mathbf{q})}] = \mathbf{q}/\rho$ is the coordinate of the vertex identified by the scalar label $\mu(\mathbf{q})$.

By using this notation, the entries of $\mathbf{G}_d$ are then given by

$$(\mathbf{G}_d)_{\nu(\boldsymbol{\ell}), \mu(\mathbf{q})} = \frac{\exp\left(-\mathrm{j}2\pi \boldsymbol{\ell}^{\mathsf{T}} \mathbf{x}_{\mu(\mathbf{q})}\right)}{\sqrt{(2M+1)^d}}$$

while its aspect ratio is

$$\beta = \frac{(2M+1)^d}{r} = \frac{(2M+1)^d}{\rho^d}. \tag{5}$$

It follows that the entries of the Hermitian Toeplitz matrix $\mathbf{T}_d = \beta \mathbf{G}_d \mathbf{G}_d^{\mathsf{H}}$ are given by

$$(\mathbf{T}_d)_{\nu(\boldsymbol{\ell}), \nu(\boldsymbol{\ell}')} = \frac{1}{\rho^d} \sum_{\mathbf{q}} \exp\left(-\mathrm{j}2\pi \mathbf{x}_{\mu(\mathbf{q})}^{\mathsf{T}} (\boldsymbol{\ell} - \boldsymbol{\ell}')\right) \tag{6}$$

where $\sum_{\mathbf{q}}$ represents a d-dimensional sum over all vectors $\mathbf{q}$ such that $q_m = 0, \dots, \rho - 1$, $m = 1, \dots, d$.

Following the approach adopted in [11], [12], in the limit for M and r growing to infinity with constant aspect ratio β and dimension d, we compute the closed form expression of $\mathbb{E}[\lambda_{d,\beta}^p]$, which can be obtained from the powers of $\mathbf{T}_d$ as [13]

$$\mathbb{E}[\lambda_{d,\beta}^p] = \int_0^\infty z^p f_\lambda(d,\beta,z)\mathrm{d}z = \lim_{\substack{M,r \xrightarrow[\beta]{} +\infty}} \frac{\mathsf{Tr}\{\mathbb{E}_{\mathcal{X}}[\mathbf{T}_d^p]\}}{(2M+1)^d}. \quad (7)$$

In (7) the symbol $\mathsf{Tr}\{\}$ identifies the matrix trace operator, and the average $\mathbb{E}_{\mathcal{X}}[\cdot]$ is computed over the set of random variables $\mathcal{X} = \{\mathbf{x}_0, \ldots, \mathbf{x}_{r-1}\}$. Notice that (7) links the moment analysis to the matrix $\mathbf{T}_d$. As an intuitive explanation consider that the eigenvalues of $\mathbf{T}_d^p$ are those of $\mathbf{T}_d$ but raised to the pth power. The trace operator performs the summation of these powers which, in the limit for the matrix size growing to infinity, is equivalent to an integration.

Using (6), the term $\mathsf{Tr}\{\mathbb{E}_{\mathcal{X}}[\mathbf{T}_d^p]\}$ in (7) can be written as

$$\begin{aligned}
\mathsf{Tr}\{\mathbb{E}_{\mathcal{X}}[\mathbf{T}_d^p]\}
&= \mathbb{E}_{\mathcal{X}}\left[\sum_{\boldsymbol{\ell}_1} (\mathbf{T}_d^p)_{\nu(\boldsymbol{\ell}_1),\nu(\boldsymbol{\ell}_1)}\right] \\
&= \mathbb{E}_{\mathcal{X}}\left[\sum_{\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_p} (\mathbf{T}_d)_{\nu(\boldsymbol{\ell}_1),\nu(\boldsymbol{\ell}_2)} \cdots (\mathbf{T}_d)_{\nu(\boldsymbol{\ell}_p),\nu(\boldsymbol{\ell}_1)}\right] \\
&= \frac{1}{r^p} \sum_{\substack{\boldsymbol{\ell}_1,\ldots,\boldsymbol{\ell}_p \\ \mathbf{q}_1,\ldots,\mathbf{q}_p}} \mathbb{E}_{\mathcal{X}}\left[\exp\left(-\mathrm{j}2\pi \sum_{i=1}^p \mathbf{x}_{\mu(\mathbf{q}_i)}^\mathsf{T}(\boldsymbol{\ell}_i - \boldsymbol{\ell}_{[i+1]})\right)\right] \\
&= \frac{1}{r^p} \sum_{\substack{\mathbf{L}\in\mathcal{L}_d \\ \mathbf{Q}\in\mathcal{Q}_d}} \mathbb{E}_{\mathcal{X}}\left[\exp\left(-\mathrm{j}2\pi \sum_{i=1}^p \mathbf{x}_{\mu(\mathbf{q}_i)}^\mathsf{T}(\boldsymbol{\ell}_i - \boldsymbol{\ell}_{[i+1]})\right)\right] \quad (8)
\end{aligned}$$

where $\mathcal{Q}_d$ and $\mathcal{L}_d$ are sets of integer matrices such that

$$\mathcal{Q}_d = \left\{\mathbf{Q} \mid \mathbf{Q} = [\mathbf{q}_1, \ldots, \mathbf{q}_p], \quad \begin{matrix} \mathbf{q}_i = [q_{i,1}, \ldots, q_{i,d}]^\mathsf{T} \\ q_{i,m} = 0, \ldots, \rho - 1 \end{matrix}\right\}$$
$$\mathcal{L}_d = \left\{\mathbf{L} \mid \mathbf{L} = [\boldsymbol{\ell}_1, \ldots, \boldsymbol{\ell}_p], \quad \begin{matrix} \boldsymbol{\ell}_i = [\ell_{i,1}, \ldots, \ell_{i,d}]^\mathsf{T} \\ \ell_{i,m} = -M, \ldots M \end{matrix}\right\}.$$

$[i+1] = i+1$, for $1 \le i < p$, and $[i+1] =$ for $i = p$. In (8), the power $\mathbf{T}_d^p$ is the product of p copies of $\mathbf{T}_d$. By substituting (6) to each of these copies, we obtain exponential terms, whose exponents are given by a sum of p terms of the form $\mathbf{x}_{\mu(\mathbf{q}_i)}^\mathsf{T}(\boldsymbol{\ell}_i - \boldsymbol{\ell}_{[i+1]})$. The average of this sum depends on the number of distinct vectors $\mathbf{q}_i$, and all possible cases can be described as *partitions* of the set $\mathcal{P} = \{1, \ldots, p\}$. In particular, the case where in the set $\{\mathbf{q}_1, \ldots, \mathbf{q}_p\}$ there are $1 \le k \le p$ distinct vectors, corresponds to a partition of $\mathcal{P}$ in k subsets. It follows that a fundamental step to calculate (7) is the computation of all possible partitions of set $\mathcal{P}$, by using a *set partitioning strategy*. Before proceeding further in our analysis, we therefore introduce some useful definitions related to set partitioning.

Definitions

Let the integer p denote the moment order and let the vector $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_p]$ be a possible combination of p integers. In our specific case, each entry of the vector $\boldsymbol{\mu}$ is given by the expression in (4), i.e., $\mu_i = \mu(\mathbf{q}_i)$ and, thus, can range between 0 and $r-1$. We define:

- the scalar integer $1 \le k(\boldsymbol{\mu}) \le p$ as the number of distinct entries of the vector $\boldsymbol{\mu}$;
- $\boldsymbol{\gamma}(\boldsymbol{\mu})$ as the vector of integers, of length $k(\boldsymbol{\mu})$, whose entries $\gamma_j(\boldsymbol{\mu})$, $j = 1, \ldots, k(\boldsymbol{\mu})$, are the entries of $\boldsymbol{\mu}$ without repetitions, in order of appearance within $\boldsymbol{\mu}$;
- $\mathcal{P}_j(\boldsymbol{\mu})$ as the set of indices of the entries of $\boldsymbol{\mu}$ with value $\gamma_j(\boldsymbol{\mu})$, $j = 1, \ldots, k(\boldsymbol{\mu})$;
- the vector $\boldsymbol{\omega}(\boldsymbol{\mu}) = [\omega_1(\boldsymbol{\mu}), \ldots, \omega_p(\boldsymbol{\mu})]$ such that, for any given $j = 1, \ldots, k(\boldsymbol{\mu})$, we have $\omega_i(\boldsymbol{\mu}) = j$ if $i \in \mathcal{P}_j(\boldsymbol{\mu})$, $i = 1, \ldots, p$.

Furthermore, we define

- Ω_p as the set of partitions of $\mathcal{P}$;
- $\Omega_{p,k}$ as the set of partitions of $\mathcal{P}$ in k subsets, $1 \le k \le p$, with $\bigcup_{k=1}^{p} \Omega_{p,k} = \Omega_p$.

Example 1: Let $\boldsymbol{\mu} = [1,\ 5,\ 2,\ 8,\ 5,\ 3,\ 2]$, then $k(\boldsymbol{\mu}) = 5$ since the entries of $\boldsymbol{\mu}$ take 5 distinct values (i.e., $\{1,\ 5,\ 2,\ 8,\ 3\}$). Such values, taken in order of appearance in $\boldsymbol{\mu}$ form the vector $\boldsymbol{\gamma}(\boldsymbol{\mu}) = [1,\ 5,\ 2,\ 8,\ 3]$. The value $\gamma_1 = 1$ appears at position 1 in $\boldsymbol{\mu}$, therefore $\mathcal{P}_1(\boldsymbol{\mu}) = \{1\}$. The value $\gamma_2 = 5$ appears at positions 2 and 5 in $\boldsymbol{\mu}$, therefore $\mathcal{P}_2(\boldsymbol{\mu}) = \{2,\ 5\}$. Similarly $\mathcal{P}_3(\boldsymbol{\mu}) = \{3,\ 7\}$, $\mathcal{P}_4(\boldsymbol{\mu}) = \{4\}$, and $\mathcal{P}_5(\boldsymbol{\mu}) = \{6\}$. By using the sets $\mathcal{P}_j$ we build the vector, $\boldsymbol{\omega}(\boldsymbol{\mu})$. For each $j = 1, \ldots, k$ we assign the value j to every ω_i such that $i \in \mathcal{P}_j$. For example, $\omega_2 = \omega_5 = 2$ since the integers 2 and 5 are in $\mathcal{P}_2$. In conclusion $\boldsymbol{\omega}(\boldsymbol{\mu}) = [1,\ 2,\ 3,\ 4,\ 2,\ 5,\ 3]$.

Note that: (i) the cardinality of Ω_p, denoted by $B(p) = |\Omega_p|$, is the pth Bell number [14] and (ii) the cardinality of $\Omega_{p,k}$, denoted by $S(p,k) = |\Omega_{p,k}|$, is a Stirling number of the second kind [15]. From the above definitions, it follows that:

1) the vector $\boldsymbol{\mu}$ induces a partition of the set $\mathcal{P}$ which is identified by the subsets $\mathcal{P}_j(\boldsymbol{\mu})$. These subsets have the properties $\bigcup_{j=1}^{k(\boldsymbol{\mu})} \mathcal{P}_j(\boldsymbol{\mu}) = \mathcal{P}$, $\mathcal{P}_j(\boldsymbol{\mu}) \cap \mathcal{P}_{j'}(\boldsymbol{\mu}) = \emptyset$ for $j \ne j'$. Even though the partition identified by $\boldsymbol{\mu}$ is often represented as $\{\mathcal{P}_1, \ldots, \mathcal{P}_{k(\boldsymbol{\mu})}\}$, by its definition, an equivalent representation of such partition is given by the vector $\boldsymbol{\omega}(\boldsymbol{\mu})$. Therefore, from now on we will refer to $\boldsymbol{\omega}(\boldsymbol{\mu})$ as a partition of the p element set $\mathcal{P}$ induced by $\boldsymbol{\mu}$ (for simplicity, however, often we will not explicit the dependency of $\boldsymbol{\omega}$ on $\boldsymbol{\mu}$);
2) $k(\boldsymbol{\omega}) = k(\boldsymbol{\mu})$, since the entries of $\boldsymbol{\omega}$ take all possible values in the set $\{1, \ldots, k(\boldsymbol{\mu})\}$;
3) $\mathcal{P}_j(\boldsymbol{\omega}) = \mathcal{P}_j(\boldsymbol{\mu})$, for $j = 1, \ldots, k(\boldsymbol{\mu})$.

At last, we define $\mathcal{M}(\boldsymbol{\omega})$ as the set of $\boldsymbol{\mu}$ inducing the same partition $\boldsymbol{\omega}$ of $\mathcal{P}$.

Example 2: Let $r = 3$ and $p = 3$. Since $\boldsymbol{\mu} = [\mu_1, \ldots, \mu_p]$ and $\mu_i = 0, \ldots, r-1$, $i = 1, \ldots, p$, we have $r^p = 27$ possible vectors $\boldsymbol{\mu}$, namely, $\{[0,\ 0,\ 0], [0,\ 0,\ 1], \ldots, [2,\ 2,\ 1], [2,\ 2,\ 2]\}$. Each $\boldsymbol{\mu}$ identifies a partition $\boldsymbol{\omega} \in \Omega_{3,k}$, with $k = 1, \ldots, 3$, as described in Example 1. The sets of partitions $\Omega_{3,k}$, are given by $\Omega_{3,1} = \{[1,\ 1,\ 1]\}$, $\Omega_{3,\,2} = \{[1,\ 1,\ 2], [1,\ 2,\ 1], [1,\ 2,\ 2]\}$, and $\Omega_{3,3} = \{[1,\ 2,\ 3]\}$, and have cardinality $S(3,\ 1) = 1$,

$S(3,\ 2) = 3$, and $S(3,\ 3) = 1$, respectively. The set of vectors $\boldsymbol{\mu}$ identifying the partition $\boldsymbol{\omega} = [1,\ 1,\ 1]$, i.e., $\mathcal{M}([1,\ 1,\ 1])$, is given by: $\mathcal{M}([1,\ 1,\ 1]) = \{[0,\ 0,\ 0],\ [1,\ 1,\ 1],\ [2,\ 2,\ 2]\}$. Similarly

$$\mathcal{M}([1,1,2]) = \left\{\begin{matrix}[0,\ 0,\ 1],\ [0,\ 0,\ 2],\ [1,\ 1,\ 0]\\ [1,\ 1,\ 2],\ [2,\ 2,\ 0],\ [2,\ 2,\ 1]\end{matrix}\right\}$$
$$\mathcal{M}([1,\ 2,\ 1]) = \left\{\begin{matrix}[0,\ 1,\ 0],\ [0,\ 2,\ 0],\ [1,\ 0,\ 1]\\ [1,\ 2,\ 1],\ [2,\ 0,\ 2],\ [2,\ 1,\ 2]\end{matrix}\right\}$$
$$\mathcal{M}([1,\ 2,\ 2]) = \left\{\begin{matrix}[0,\ 1,\ 1],\ [0,\ 2,\ 2],\ [1,\ 0,\ 0]\\ [1,\ 2,\ 2],\ [2,\ 0,\ 0],\ [2,\ 1,\ 1]\end{matrix}\right\}$$
$$\mathcal{M}([1,\ 2,\ 3]) = \left\{\begin{matrix}[0,\ 1,\ 2],\ [0,\ 2,\ 1],\ [1,\ 0,\ 2]\\ [1,\ 2,\ 0],\ [2,\ 0,\ 1],\ [2,\ 1,\ 0]\end{matrix}\right\}.$$

B. Closed Form Expression of $\mathbb{E}[\lambda^p_{d,\beta}]$

By using the definitions in Section III-B and by applying set partitioning strategy to (8), we can state our first main result.

Theorem 3.1: Let $\mathbf{T}_d$ be a $(2M+1)^d \times (2M+1)^d$ Hermitian random matrix as defined in (6), where the properties of the random vectors $\mathbf{x}_{\mu(\mathbf{q})}$ are described in Section III-A. Then, for any given β and d, the pth moment of the asymptotic eigenvalue distribution of $\mathbf{T}_d$ is given by

$$\mathbb{E}[\lambda^p_{d,\beta}] = \sum_{k=1}^{p}\sum_{h=1}^{k}\beta^{p-h}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}\sum_{\boldsymbol{\omega}'\in\Omega_{k,h}} u(\boldsymbol{\omega}')v(\boldsymbol{\omega},\boldsymbol{\omega}')^d \quad (9)$$

where $u(\boldsymbol{\omega}') = (-1)^{k-h}\prod_{j'=1}^{h}(|\mathcal{P}_{j'}(\boldsymbol{\omega}')|-1)!$ and where

$$v(\boldsymbol{\omega},\boldsymbol{\omega}') = \begin{cases}\int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\mathrm{d}\mathbf{y} & h=1\\ \int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\mathbf{y} & {h>1 \atop h<k}\\ \int_{\mathcal{H}_p}\prod_{j=1}^{k}\delta_D\left(w_j(\boldsymbol{\omega})\right)\mathrm{d}\mathbf{y} & h=k.\end{cases} \quad (10)$$

In (10) we defined

$$\tilde{C}_j(\boldsymbol{\omega}) = C\left(-\mathrm{j}2\pi\beta^{1/d}w_j(\boldsymbol{\omega})\right) \quad (11)$$

$$\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}') = \delta_D\left(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')} w_{i'}(\boldsymbol{\omega})\right) \quad (12)$$

the set $\mathcal{H}_p$ as the p-dimensional hypercube $[-1/2,\ 1/2)^p$, $C(s) = \mathbb{E}_{\tilde{x}}[\mathrm{e}^{sz}]$ as the characteristic function of $\tilde{x}$, and $\delta_D(\cdot)$ as the Dirac's delta. Moreover $w_j(\boldsymbol{\omega}) = \sum_{i\in\mathcal{P}_j(\boldsymbol{\omega})} y_i - y_{[i+1]}$, $y_i \in \mathbb{R}$, $i = 1,\ldots,p$, and $j = 1,\ldots,k(\boldsymbol{\omega})$. In particular, for $k=1$, we have $v(\boldsymbol{\omega},\boldsymbol{\omega}') = 1$.

Proof: The proof can be found in Appendix A. ■

With the aim to give an intuitive explanation of the above expressions, note that the right-hand side (RHS) of (9) counts all possible partitions of the set $\mathcal{P} = \{1,\ldots,p\}$, $C(s)$ in (10) accounts for the generic distribution of the variables $\tilde{\mathbf{x}}$, and the quantity $w_j(\boldsymbol{\omega})$ represents the indices pairing that appears in the exponent of the generic entry of the power $\mathbf{T}^p_d$.

TABLE I
PARTITION SETS $\Omega_{n,m}$ FOR $n = 1,\ 2,\ 3$, AND $1 \le m \le n$. EACH PARTITION IS REPRESENTED THROUGH ITS ASSOCIATED VECTOR $\boldsymbol{\omega}$ AND THE VALUE OF $u(\boldsymbol{\omega})$

$\boldsymbol{\omega}, u(\boldsymbol{\omega})$	$m=1$	$m=2$	$m=3$
$n=1$	[1], 1		
$n=2$	[1,1], -1	[1,2], 1	
$n=3$	[1,1,1], 2	[1,1,2], -1 [1,2,1], -1 [1,2,2], -1	[1,2,3], 1

To further clarify the moments computation, Table I reports an example of partition sets $\Omega_{n,m}$ for $n = 1,\ldots,3$ and $1 \le m \le n$, while Example 3 shows the computation of the second moment of the eigenvalue distribution.

1) Example 3: We compute the analytic expression of $\mathbb{E}[\lambda^2_{d,\beta}]$. Using (9), we get

$$\mathbb{E}[\lambda^2_{d,\beta}] = \sum_{k=1}^{2}\sum_{h=1}^{k}\beta^{2-h}\sum_{\boldsymbol{\omega}\in\Omega_{2,k}}\sum_{\boldsymbol{\omega}'\in\Omega_{k,h}} u(\boldsymbol{\omega}')v(\boldsymbol{\omega},\boldsymbol{\omega}')^d.$$

By expanding this expression and using Table I, we obtain $\mathbb{E}[\lambda^2_{d,\beta}] = \beta v([1,\ 1],\ [1])^d - \beta v([1,\ 2],\ [1,\ 1])^d + v([1,\ 2],[1,\ 2])^d$. By using (10) we have $v([1,\ 1],\ [1]) = 1$, $v([1,\ 2],[1,\ 2]) = 1$, and

$$v([1,\ 2],\ [1,\ 1]) = \int_{\mathcal{H}_2}\left|C\left(-\mathrm{j}2\pi\beta^{1/d}(y_1-y_2)\right)\right|^2\mathrm{d}\mathbf{y}.$$

IV. CONVERGENCE TO THE MARČENKO-PASTUR DISTRIBUTION

Here we show that the asymptotic eigenvalue distribution of the matrix $\mathbf{T}_d$ tends to the Marčenko-Pastur law [10], as $d \to \infty$, i.e., $\lim_{d\to\infty} f_\lambda(d,\beta,z) = f_{\mathrm{MP}}(\beta,z) = \sqrt{(c_1-z)(z-c_2)}/(2\pi z\beta)$ where $c_1, c_2 = (1 \pm \sqrt{\beta})^2$, $0 < \beta \le 1$, $c_2 \le x \le c_1$. This is equivalent to prove that, as $d \to \infty$, the pth moment of $\lambda_{d,\beta}$ tends to the pth moment of the Marčenko-Pastur distribution with parameter β, for every $p \ge 1$.

Theorem 4.1: Let $\mathbf{T}_d$ be a $(2M+1)^d \times (2M+1)^d$ Hermitian random matrix as defined in (6), where the properties of the random vectors $\mathbf{x}_{\mu(\mathbf{q})}$ are described in Section I-A. Let $\mathbb{E}[\lambda^p_{d,\beta}]$ be the pth moment of the asymptotic eigenvalue distribution of $\mathbf{T}_d$, given by Theorem 3.1. Then, for any given β

$$\lim_{d\to\infty}\mathbb{E}[\lambda^p_{d,\beta}] = \mathbb{E}[\lambda^p_{\infty,\beta}] = \sum_{k=1}^{p}\beta^{p-k}N(p,k)$$

where $N(p,k)$ are the *Narayana numbers* [16], [17] and $\mathbb{E}[\lambda^p_{\infty,\beta}]$ are the *Narayana polynomials*, i.e., the moments of the Marčenko-Pastur distribution [10].

Proof: We first look at the expression of the pth asymptotic moment and observe that, for $h = k$, the contribution of the term in the RHS of (9) reduces to

$$\sum_{k=1}^{p}\beta^{p-k}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}\sum_{\boldsymbol{\omega}'\in\Omega_{k,k}} u(\boldsymbol{\omega}')v(\boldsymbol{\omega},\boldsymbol{\omega}')^d. \quad (13)$$

The cardinality of $\Omega_{k,k}$ is $S(k,k) = 1$ and $\Omega_{k,k} = \{[1,\ldots,k]\}$. Thus, we only consider $\boldsymbol{\omega}' = [1,\ldots,k]$. Moreover, using the definition of $u(\boldsymbol{\omega}')$ after (10) we have $u([1,\ldots,k]) = 1$ since each subset $\mathcal{P}_{j'}([1,\ldots,k])$ has cardinality 1, $j' = 1,\ldots,k$. Therefore, the term in (13) becomes $\sum_{k=1}^{p}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}\beta^{p-k}v(\boldsymbol{\omega},[1,\ldots,k])^d$. Using (10) for $h = k$ we have: $v(\boldsymbol{\omega},[1,\ldots,k]) = \int_{\mathcal{H}_p}\prod_{j=1}^{k}\delta_D(w_j(\boldsymbol{\omega}))\,\mathrm{d}\mathbf{y} \triangleq v(\boldsymbol{\omega})$. Hence, the contribution to the pth moment reduces to

$$\sum_{k=1}^{p}\beta^{p-k}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}v(\boldsymbol{\omega})^d. \tag{14}$$

In [3] it is shown that, as $d\longrightarrow\infty$, (14) tends to the Narayana polynomial of order p. Thus, to prove the theorem, it is enough to show that, for $h < k$, the contribution of the term in the RHS of (9), to the expression of the pth asymptotic moment, vanishes as $d\longrightarrow\infty$. For example, we have to show that, for each $\boldsymbol{\omega}\in\Omega_{p,k}$ and $\boldsymbol{\omega}'\in\Omega_{k,h}$, with $h < k$

$$\lim_{d\longrightarrow\infty} v(\boldsymbol{\omega},\,\boldsymbol{\omega}')^d = 0 \tag{15}$$

or, equivalently, that $|v(\boldsymbol{\omega},\boldsymbol{\omega}')| < 1$.

By using (10), we first notice that for $1 < h < k$

$$\begin{aligned}|v(\boldsymbol{\omega},\boldsymbol{\omega}')| &= \left|\int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega}',\boldsymbol{\omega}')\mathrm{d}\mathbf{y}\right|\\ &\le \int_{\mathcal{H}_p}\left|\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')\right|\mathrm{d}\mathbf{y}\\ &= \int_{\mathcal{H}_p}\prod_{j=1}^{k}\left|\tilde{C}_j(\boldsymbol{\omega})\right|\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\mathbf{y}\end{aligned} \tag{16}$$

since by (12)$\tilde{D}_{j'}(\boldsymbol{\omega},\,\boldsymbol{\omega}') > 0$. Moreover, from (11) and the definition of the characteristic function of $\tilde{x}$, we have

$$\begin{aligned}\left|\tilde{C}_j(\boldsymbol{\omega})\right| &= \left|C\left(-\mathrm{j}2\pi\beta^{1/d}w_j(\boldsymbol{\omega})\right)\right|\\ &= \left|\int_{-\infty}^{+\infty}\exp\left(-\mathrm{j}2\pi\beta^{1/d}w_j(\boldsymbol{\omega})z\right)f_{\tilde{x}}(z)\mathrm{d}z\right|\\ &\overset{(a)}{\le}\int_{-\infty}^{+\infty}\left|\exp\left(-\mathrm{j}2\pi\beta^{1/d}w_j(\boldsymbol{\omega})z\right)f_{\tilde{x}}(z)\right|\mathrm{d}x\\ &= \int_{-\infty}^{+\infty}f_{\tilde{x}}(z)\mathrm{d}z = 1.\end{aligned} \tag{17}$$

Next, we make the following observations:

i) the equality (a) in (17) arises when $w_j(\boldsymbol{\omega}) = 0$, else if $w_j(\boldsymbol{\omega}) \ne 0$, the strict inequality $|\tilde{C}_{j'}(\boldsymbol{\omega})| < 1$ holds;

ii) since we consider partitions $\boldsymbol{\omega}'$ of the set $\{1,\ldots,k\}$ in h subsets $\mathcal{P}_{j'}(\boldsymbol{\omega}')$, $j' = 1,\ldots,h$, with $h < k$, then at least one of the sets $\mathcal{P}_{j'}(\boldsymbol{\omega}')$ has cardinality $|\mathcal{P}_{j'}(\boldsymbol{\omega}')| > 1$;

iii) from the definition in (12) the term $\tilde{D}_{j'}(\boldsymbol{\omega},\,\boldsymbol{\omega}')$ in (16) gives a nonzero contribution to the integral in (16) only when $\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}w_{i'}(\boldsymbol{\omega}) = 0$, for every $j' = 1,\ldots,h$. The number of terms in the argument of the $\delta_D(\cdot)$ function equals $|\mathcal{P}_{j'}(\boldsymbol{\omega}')|$, for every j'.

Thus, if for some j', $|\mathcal{P}_{j'}(\boldsymbol{\omega}')| > 1$, the corresponding arguments of the $\delta_D(\cdot)$ function will contain two or more terms, whose sum need to be zero in order to provide a nonzero contribution to (16). By consequence, there always exist some $w_{i'}(\boldsymbol{\omega}') \ne 0$ providing a nonzero contribution to the integral in (16). Therefore, by using observation i) the strict inequality $|\tilde{C}_j(\boldsymbol{\omega})| < 1$ always holds for at least an integer $j\in\{1,\ldots,h\}$. We can then write $|v(\boldsymbol{\omega},\boldsymbol{\omega}')| < \int_{\mathcal{H}_p}\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\mathbf{y} \le 1$ which proves the claim (15).

When $h = 1$, again, there is a measurable subset of $\mathcal{H}_p$ for which $w_j(\boldsymbol{\omega}) \ne 0$, hence, $|v(\boldsymbol{\omega},\boldsymbol{\omega}')| \le \int_{\mathcal{H}_p}\prod_{j=1}^{k}|\tilde{C}_j(\boldsymbol{\omega})|\mathrm{d}\mathbf{y} < 1$ i.e., the strict inequality holds and (15) is proved. ■

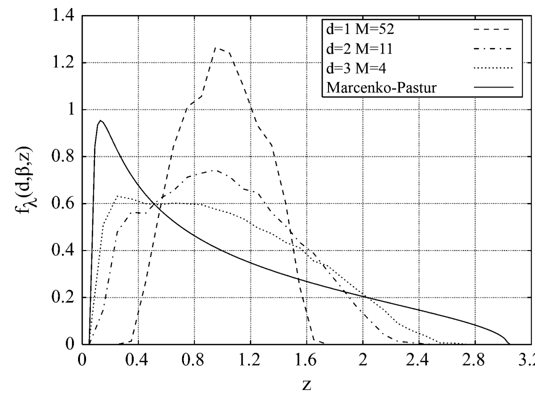

Fig. 1. Comparison between the Marčenko-Pastur distribution and the empirical distribution for $\beta = 0.55$ and $d = 1,\ 2,\ 3$ in the quasi equally space case, and uniform $f_{\tilde{x}}(z)$.

In Fig. 1, we show the empirical eigenvalue distribution of the matrix $\mathbf{T}_d$ for $\beta = 0.55$, $d = 1,2,3$, and $\tilde{x}$ uniformly distributed in $[0,\ 1]$. The empirical distribution is compared to the Marčenko-Pastur distribution (solid line). We observe that as, d increases, the Marčenko-Pastur distribution law becomes a good approximation of $f_\lambda(d,\beta,z)$. In particular, the two curves are relatively close for small z, already for $d = 3$. Curves for $d > 3$ are not shown since for large d numerically deriving the eigenvalue distribution leads to cumbersome computations.

V. Applications

Here we present some applications where the results derived in this work can be used.

The closed form expression of the moments of $f_\lambda(d,\beta,z)$, given by (9), can be a useful basis for performing deconvolution operations, as proposed in [6]. As for the asymptotic approximation, we show below how to exploit our results for the estimation of the mean square error (MSE) provided by linear reconstruction techniques of irregularly sampled signals.

Let us assume a general linear system model affected by additive noise. For simplicity, consider a one-dimensional signal, $s(x)$. When observed over a finite interval, it admits an infinite Fourier series expansion [1]. We can think of the largest index M of the nonnegligible Fourier coefficients of the expansion as the approximate one-sided bandwidth of the signal. We therefore represent $s(x)$ by using $2M+1$ complex harmonics as

$$s(x) = \frac{1}{\sqrt{2M+1}}\sum_{k=-M}^{M}a_\ell\exp\left(\mathrm{j}2\pi\ell x\right).$$

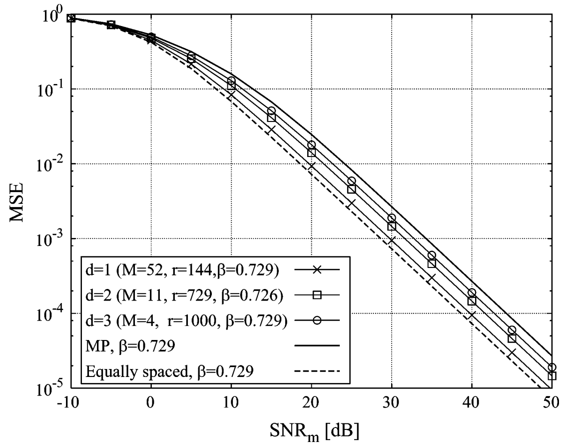

Fig. 2. MSE as a function of the signal-to-noise ratio for $d = 1, 2, 3$. The curves are compared with the results obtained through our asymptotic analysis (MP) and with the equally spaced case.

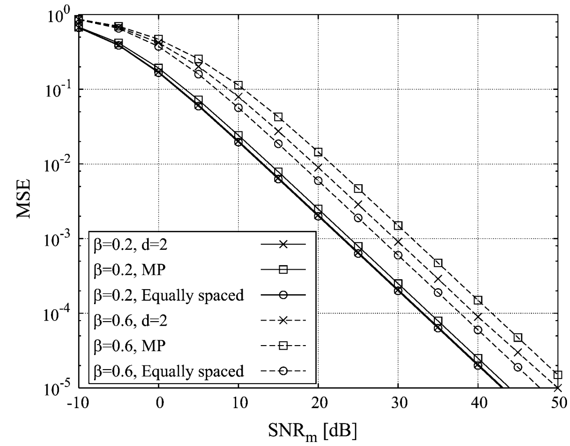

Fig. 3. MSE as a function of the signal-to-noise ratio for $\beta = 0.2, 0.6$. The curves are obtained for $d = 2$ and compared against both the equally spaced case and the results derived through our asymptotic analysis (MP).

Now, consider that the signal is observed within one period interval $[0, 1)$ and sampled in r points placed at positions $\mathbf{x} = [x_0, \dots, x_{r-1}]^\mathsf{T}$, $x_q \in [0, 1)$, $q = 0, \dots, r-1$. The complex numbers a_ℓ represent amplitudes and phases of the harmonics in $s(x)$. The signal samples $\mathbf{s} = [s(x_0), \dots, s(x_{r-1})]^\mathsf{T}$ can be written as $\mathbf{s} = \mathbf{G}_1^\mathsf{H}\mathbf{a}$, where the matrix $\mathbf{G}_1$ is given in (2). The signal discrete spectrum is given by the $2M+1$ complex vector $\mathbf{a} = [a_{-M}, \dots, a_0, \dots, a_M]^\mathsf{T}$. We can now write the linear model for a measurement sample vector $\mathbf{p} = [p(x_0), \dots, p(x_{r-1})]^\mathsf{T}$ taken at the sampling points x_q as

$$\mathbf{p} = \mathbf{s} + \mathbf{n} = \mathbf{G}_1^\mathsf{H}\mathbf{a} + \mathbf{n}$$

where $\mathbf{n}$ is a random vector representing measurement noise. The general problem is to reconstruct $\mathbf{s}$ or $\mathbf{a}$ given the noisy measurements $\mathbf{p}$[3]. A commonly used parameter to measure the quality of the estimate of the reconstructed signal is the MSE. In [1] and [2], it has been shown that, when linear reconstruction techniques are used and the sample coordinates are known, the asymptotic MSE (i.e., as the number of harmonics and the number of samples tend to infinity while their ratio is kept constant) is a function of the asymptotic eigenvalue distribution of the matrix $\mathbf{T}_1 = \beta\mathbf{G}_1\mathbf{G}_1^\mathsf{H}$, i.e.,

$$\mathrm{MSE} = \mathbb{E}_{\lambda}\left[\frac{\beta}{\lambda\,\mathrm{SNR}_m + \beta}\right] \tag{18}$$

where the random variable λ has distribution $f_\lambda(1, \beta, z)$ and SNR_m is the signal-to-noise ratio on the measure. Expression (18) also hold for the d-dimensional case, with the pdf of λ given by $f_\lambda(d, \beta, z)$. We therefore exploit our asymptotic approximation to $f_\lambda(d, \beta, z)$ to compute (18).

Fig. 2 shows the MSE obtained as a function of the signal-to-noise ratio SNR_m. The curves with markers labeled by "$d = 1, 2, 3$" refer to the cases where the signal has dimension d and the sampling points are quasi-equally spaced with jitter $\tilde{x}$, uniformly distributed over $[0, 1)$, and $\beta = 0.729$. The curve labeled by "MP" (thick line) reports the results derived through our asymptotic ($d \longrightarrow \infty$) approximation to the eigenvalue distribution. The curve labeled by "Equally spaced" (dashed line) represents the MSE achieved under a perfect equally spaced sample placement, i.e., when the jitters $\tilde{\mathbf{x}}_q$ described in Section III-A are given by $\tilde{\mathbf{x}}_q = \mathbf{0}$, for $q = 0, \dots, r-1$. In such case it is straightforward to show that $\mathbf{T}_d$ is the identity matrix and that its eigenvalue distribution is given by $f_\lambda(d, \beta, z) = \delta_D(z-1)$. Notice that in Fig. 2 the MSE grows as d increases and tends to the MSE obtained by a Marčenko-Pastur eigenvalue distribution. Instead, as expected, the "Equally spaced" curve represents a lower bound to the system performance.

Fig. 3 presents similar results but obtained for $d = 2$ and different values of β. We observe that the MSE obtained through our asymptotic approximation (the curve labeled by "MP") gives excellent results for values of β as small as 0.2, even when compared against the numerical results derived by fixing $d = 2$. For $\beta = 0.6$ (i.e., when the ratio of the number of signal harmonics to the number of samples increases), the approximation becomes slightly looser, and the MSE computed by using the Marčenko-Pastur distribution gives an upper limit to the quality of the reconstructed signal. Note that the smaller the β, the higher the oversampling rate relative to the equally spaced minimal sampling rate $\beta = 1$. We thus observe how our bound becomes tighter as the oversampling rate increases.

To conclude, we describe some areas in signal processing where the above system model and results find application.

i) *Spectral estimation with noise*. Spectral estimation from high precision sampling and quantization of bandlimited signals uses measurement systems which are usually affected by jitter [18]. In such applications the quantization noise corresponds to the measurement noise and the jitter is caused by the limited accuracy of the timing circuits. In this case the sampling points are mismatched with respect to the nominal values, thus for $d = 1$ we have: $x_q = q/r + \tilde{x}_q/r$ with some sampling rate $1/r$. Note that the exact positions of the samples are not known and the case studied in this paper (i.e., MSE with exact positions) gives a lower bound to the reconstruction error.

ii) *Signal reconstruction in sensor networks*. Sensor networks, whose nodes sample a physical field, like air temperature, light intensity, pollution levels or rain falls, typically represent an example of quasi-equally spaced sampling [2], [7], [19], [20]. Indeed, often sensors are not regularly deployed in the area of interest due to terrain conditions and deployment practicality and, thus, the physical field is not regularly sampled in the space domain. Sensors report the data to a common processing unit (or *sink* node), which is in charge of reconstructing the sensed field, based on the received samples and on the knowledge of their coordinates. If the field can be approximated as bandlimited in the space domain, then an estimate of the discrete spectrum can be obtained by using linear reconstruction techniques [2], [21], even in presence of additive noise. In this case, our approximation allows to compute the MSE on the reconstructed field.

iii) *Stochastic sampling in computer graphics and image processing*. Jittered sampling was first examined by Balakrishnan in [22], who analyzed it as an undesirable effect in sampling continuous time functions. More than twenty years later, Cook [23] realized that the effect of stochastic sampling can be advantageous in computer graphics to reduce aliasing artifacts, and considered jittering a regular grid as an effective sampling technique. Another example of sampling with jitter was recently proposed in [24], for robust authentication of images.

VI. Conclusions

We studied the behavior of the eigenvalue distribution of a class of random matrices, which find large application in signal and image processing. In particular, by using asymptotic analysis, we derived a closed-form expression for the moments of the eigenvalue distribution. Using these moments, we showed that, as the signal dimension goes to infinity, the asymptotic eigenvalue distribution tends to the Marčenko-Pastur law. This result allowed us to obtain a simple and accurate bound to the signal reconstruction error, which can find application in several fields, such as jittered sampling, sensor networks, computer graphics and image processing.

Appendix A
Table of Symbols

Random variables and distributions

$f_v(z)$	pdf of the random variable v
$\mathsf{F}_v(z)$	cdf of the random variable v
$C(s)$	characteristic function of the random variable $\tilde{x}$

Random matrices

$\mathbf{G}_d$	random matrix of size $(2M+1)^d \times r$
β	aspect ratio of matrix $\mathbf{G}_d$, equal to $(2M+1)^d/r$
$\mathbf{T}_d$	Hermitian Toeplitz matrix given by $\beta\mathbf{G}_d\mathbf{G}_d^{\mathsf{H}}$
$f_\lambda(d,\beta,z)$	asymptotic eigenvalue distribution of $\mathbf{T}_d$
$\mathbb{E}[\lambda_{d,\beta}^p]$	pth moment of $f_\lambda(d,\beta,z)$
$\mu(\mathbf{q})$	function mapping the vertex coordinate $\mathbf{q}$ to the column index of $\mathbf{G}_d$
$\nu(\boldsymbol{\ell})$	function mapping the vector of integers $\boldsymbol{\ell}$ to the row index of $\mathbf{G}_d$

Set partitioning

$\mathcal{P}$	set of integers $\{1,\ldots,p\}$
Ω_p	set of partitions of $\mathcal{P}$
$\Omega_{p,k}$	set of partitions of $\mathcal{P}$ in k subsets
Ω_k	set of partitions of $\{1,\ldots,k\}$
$\Omega_{k,h}$	set of partitions of $\{1,\ldots,k\}$ in h subsets
$\boldsymbol{\omega}$	partition, $\boldsymbol{\omega}\in\Omega_p$
$\boldsymbol{\omega}'$	partition, $\boldsymbol{\omega}'\in\Omega_k$
$\mathcal{P}_{j'}(\boldsymbol{\omega}')$	set of indices i such that $\omega'_i = j$
$\mathcal{M}(\boldsymbol{\omega})$	set of vectors $\boldsymbol{\mu}$ inducing the partition $\boldsymbol{\omega}$

Appendix B
Set Partitioning

To prove Theorem 3.1, first we apply the definitions in Section III-B to rewrite (8) using set partitioning. In particular, by considering the vector $\boldsymbol{\mu} = \boldsymbol{\mu}(\mathbf{Q}) \triangleq [\mu_1,\ldots,\mu_p]^{\mathsf{T}}$ where $\mu_i = \mu(\mathbf{q}_i)$ and $\mathbf{q}_i$ is the ith column of $\mathbf{Q}$, we observe the following:

- the vector $\boldsymbol{\mu}$ is uniquely defined by $\mathbf{Q}$, and a given $\boldsymbol{\mu}$ uniquely defines a matrix $\mathbf{Q}\in\mathcal{Q}_d$ since $\mu(\cdot)$ is an invertible function;
- a given $\boldsymbol{\mu}$ induces a partition $\boldsymbol{\omega}(\boldsymbol{\mu})$;
- since r is the number of values that the entries μ_i can take, there exist $r!/(r-k(\boldsymbol{\mu}))!$ matrices $\mathbf{Q}\in\mathcal{Q}_d$ generating a given partition of $\mathcal{P}$ made of $k(\boldsymbol{\mu})$ subsets. In other words $r!/(r-k(\boldsymbol{\mu}))!$ distinct $\boldsymbol{\mu}$'s yield the same partition $\boldsymbol{\omega}(\boldsymbol{\mu})$.

Since the random vectors $\mathbf{x}_{\mu(\mathbf{q}')}$ and $\mathbf{x}_{\mu(\mathbf{q}'')}$ are independent for $\mathbf{q}'\neq\mathbf{q}''$, for any given $\mathbf{Q}$ the average operator in (8) factorizes into $k(\boldsymbol{\mu})$ terms, i.e.,

$$\begin{aligned}\mathop{\mathbb{E}}_{\mathcal{X}}&\left[\exp\left(-\mathrm{j}2\pi\sum_{i=1}^{p}\mathbf{x}_{\mu(\mathbf{q}_i)}^{\mathsf{T}}(\boldsymbol{\ell}_i-\boldsymbol{\ell}_{[i+1]})\right)\right]\\&=\mathop{\mathbb{E}}_{\mathcal{X}}\left[\exp\left(-\mathrm{j}2\pi\sum_{i=1}^{p}\mathbf{x}_{\mu_i}^{\mathsf{T}}(\boldsymbol{\ell}_i-\boldsymbol{\ell}_{[i+1]})\right)\right]\\&=\prod_{j=1}^{k(\boldsymbol{\mu})}\mathop{\mathbb{E}}_{\mathbf{x}_{\gamma_j}}\left[\zeta^{\rho\mathbf{x}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right]\end{aligned}\tag{19}$$

indeed, for every $i\in\mathcal{P}_j(\boldsymbol{\mu})$, we have $\mu_i=\gamma_j$. In the last line of (19), we defined $\zeta=\exp(-\mathrm{j}2\pi/\rho)$ and

$$\hat{\mathbf{w}}_j(\boldsymbol{\mu})=\sum_{i\in\mathcal{P}_j(\boldsymbol{\mu})}\boldsymbol{\ell}_i-\boldsymbol{\ell}_{[i+1]}.\tag{20}$$

Also, note that, in the product in (19), each factor depends on a single random vector, $\mathbf{x}_{\gamma_j}$. Since $\mathbf{x}_{\mu(\mathbf{q})}=\mathbf{q}/\rho+\tilde{\mathbf{x}}_{\mu(\mathbf{q})}/\rho$

and $\mu(\cdot)$ is invertible then, by defining $\bar{\mathbf{x}}_{\gamma_j} = \mu^{-1}(\gamma_j)$ we have $\mathbf{x}_{\gamma_j} = \bar{\mathbf{x}}_{\gamma_j}/\rho + \tilde{\mathbf{x}}_{\gamma_j}/\rho$ and

$$\mathop{\mathbb{E}}_{\mathbf{x}_{\gamma_j}}\left[\zeta^{\rho\mathbf{x}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right] = \zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}_{\gamma_j}}\left[\zeta^{\tilde{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right] = \zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right]. \quad (21)$$

In the last term of (21) we removed the subscript γ_j from the argument of the average operator, since the distribution of $\tilde{\mathbf{x}}_{\gamma_j}$ does not depend on γ_j. Summarizing, the term $\mathsf{Tr}\{\mathop{\mathbb{E}}_{\mathcal{X}}[\mathbf{T}_d^p]\}$ in (7) can be written as

$$\mathsf{Tr}\{\mathop{\mathbb{E}}_{\mathcal{X}}[\mathbf{T}_d^p]\} = \frac{1}{r^p}\sum_{\substack{\mathbf{Q}\in\mathcal{Q}_d\\ \mathbf{L}\in\mathcal{L}_d}}\prod_{j=1}^{k(\boldsymbol{\mu})}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right].$$

Since each $\mathbf{Q}$ is uniquely identified by a vector $\boldsymbol{\mu}$, we can observe that

$$\sum_{\mathbf{Q}\in\mathcal{Q}_d} f(\boldsymbol{\mu}) = \sum_{\boldsymbol{\omega}\in\Omega_p}\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})} f(\boldsymbol{\mu}) = \sum_{k=1}^{p}\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})} f(\boldsymbol{\mu})$$

for every function $f(\boldsymbol{\mu})$. $\mathcal{M}(\boldsymbol{\omega})$ represents the set of $\boldsymbol{\mu}$ inducing a given partition $\boldsymbol{\omega}$.

From the definitions in Section III-B, it follows that, if $\boldsymbol{\mu}$ induces $\boldsymbol{\omega}$, then $k(\boldsymbol{\mu}) = k(\boldsymbol{\omega})$, $\mathcal{P}_j(\boldsymbol{\mu}) = \mathcal{P}_j(\boldsymbol{\omega})$, and $\hat{\mathbf{w}}_j(\boldsymbol{\mu}) = \hat{\mathbf{w}}_j(\boldsymbol{\omega})$, $j = 1,\ldots,k(\boldsymbol{\omega})$. Therefore,

$$\begin{aligned}\mathsf{Tr}\{\mathop{\mathbb{E}}_{\mathcal{X}}[\mathbf{T}_d^p]\} &= \frac{1}{r^p}\sum_{k=1}^{p}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})\\ \mathbf{L}\in\mathcal{L}_d}}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\mu})}\right]\\ &= \frac{1}{r^p}\sum_{k=1}^{p}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})\\ \mathbf{L}\in\mathcal{L}_d}}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}\right]\\ &= \frac{1}{r^p}\sum_{k=1}^{p}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \mathbf{L}\in\mathcal{L}_d\\ \boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}}\left[\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}\right]\prod_{j=1}^{k}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}\right]\\ &\overset{(a)}{=} \frac{1}{r^p}\sum_{k=1}^{p}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \mathbf{L}\in\mathcal{L}_d}}\eta(\boldsymbol{\omega},\mathbf{L})\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}.\end{aligned} \quad (22)$$

In (22) we defined

$$\eta(\boldsymbol{\omega},\mathbf{L}) = \prod_{j=1}^{k}\mathop{\mathbb{E}}_{\tilde{\mathbf{x}}}\left[\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}\right] = \prod_{j=1}^{k}\prod_{m=1}^{d}\mathop{\mathbb{E}}_{\tilde{x}_m}\left[\zeta^{\tilde{x}_m\hat{w}_{jm}(\boldsymbol{\omega})}\right] \quad (23)$$

where $\tilde{x}_m$ and $\hat{w}_{jm}$ are the mth entries of $\tilde{\mathbf{x}}$ and $\hat{\mathbf{w}}_j$, respectively. In the equality "(a)" we exploited the fact that the term $\zeta^{\tilde{\mathbf{x}}^{\mathsf{T}}\hat{\mathbf{w}}_j(\boldsymbol{\omega})}$ does not depend on $\boldsymbol{\mu}$ and can be factored from the sum over $\boldsymbol{\mu}$. As for the term $\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j}$, we have the following lemma.

Lemma B.1: Let $\boldsymbol{\omega}\in\Omega_{p,k}$, let $\hat{\mathbf{w}}_1,\ldots,\hat{\mathbf{w}}_k$ be vectors of size d with integer entries, defined as in (20). Let $\mathcal{M}(\boldsymbol{\omega})$ be the set of vectors $\boldsymbol{\mu}$ inducing $\boldsymbol{\omega}$. Then

$$\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j} = \sum_{h=1}^{k} r^h\sum_{\boldsymbol{\omega}'\in\Omega_{k,h}} u(\boldsymbol{\omega}')\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}')$$

where $\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}') = \prod_{j'=1}^{h}\delta_K(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}\hat{\mathbf{w}}_{i'}(\boldsymbol{\omega}))$ and $\delta_K(\cdot)$ is the Kronecker's delta function. Moreover $u(\boldsymbol{\omega}') = (-1)^{k-h}\prod_{j'=1}^{h}(|\mathcal{P}_{j'}(\boldsymbol{\omega}')|-1)!$, $\gamma_j = \gamma_j(\boldsymbol{\mu})$, and $\Omega_{k,h}$ is the set of vectors $\boldsymbol{\omega}'$ of size k, representing the partitions of the set $\mathcal{P}' = \{1,\ldots,k\}$ in h subsets, namely, $\mathcal{P}'_1(\boldsymbol{\omega}'),\ldots,\mathcal{P}'_h(\boldsymbol{\omega}')$.

Proof: The proof can be found in Appendix C. ■

By applying the result of Lemma B.1 to (22), we get

$$\mathsf{Tr}\{\mathop{\mathbb{E}}_{\mathcal{X}}[\mathbf{T}_d^p]\} = \sum_{\substack{h=1,\ldots,k\\ k=1,\ldots,p\\ \boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\omega}'\in\Omega_{k,h}}}\frac{u(\boldsymbol{\omega}')}{r^{p-h}}\sum_{\mathbf{L}\in\mathcal{L}_d}\eta(\boldsymbol{\omega},\mathbf{L})\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}'). \quad (24)$$

Since $\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}') = \prod_{j'=1}^{h}\prod_{m=1}^{d}\delta_K(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}\hat{w}_{i'm}(\boldsymbol{\omega}))$ by using (23) we have

$$\begin{aligned}\sum_{\mathbf{L}\in\mathcal{L}_d}\eta(\boldsymbol{\omega},\mathbf{L})\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}') &= \Bigg[\sum_{\boldsymbol{\ell}\in\mathcal{L}_1}\prod_{j=1}^{k}\mathbb{E}_{\tilde{x}}\left[\zeta^{\tilde{x}\hat{w}_j(\boldsymbol{\omega})}\right]\\ &\quad\times\prod_{j'=1}^{h}\delta_K\Bigg(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}\hat{w}_{i'}(\boldsymbol{\omega})\Bigg)\Bigg]^d\\ &= \psi_M(\boldsymbol{\omega},\boldsymbol{\omega}')^d\end{aligned} \quad (25)$$

where the subscript M of the function ψ highlights the dependency of $\boldsymbol{\ell}$ on M. From (25) we note that $\mathbb{E}_{\tilde{x}}[\zeta^{\tilde{x}\hat{w}_j(\boldsymbol{\omega})}] = C(-\mathrm{j}2\pi\hat{w}_j(\boldsymbol{\omega})/\rho)$ where $C(s) = \mathop{\mathbb{E}}_{\tilde{x}}[\mathrm{e}^{sz}]$ is the characteristic function of $\tilde{x}$. Moreover, by using (5), we observe that $1/r^{p-k} = \beta^{p-h}/(2M+1)^{d(p-h)}$. In conclusion, we compute $\mathbb{E}[\lambda_{d,\beta}^p]$, by evaluating the limit in (7). To this end, we use (25), and (24) in (7), and we obtain

$$\begin{aligned}\mathbb{E}[\lambda_{d,\beta}^p] &= \lim_{M,r\xrightarrow[\beta]{}+\infty}\sum_{\substack{k=1,\ldots,p\\ h=1,\ldots,k}}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\omega}'\in\Omega_{k,h}}}\times\frac{\beta^{p-h}u(\boldsymbol{\omega}')\psi_M(\boldsymbol{\omega},\boldsymbol{\omega}')^d}{(2M+1)^{d(p-h+1)}}\\ &= \sum_{\substack{k=1,\ldots,p\\ h=1,\ldots,k}}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\omega}'\in\Omega_{k,h}}}\beta^{p-h}u(\boldsymbol{\omega}')\times\left[\lim_{M\to\infty}\frac{\psi_M(\boldsymbol{\omega},\boldsymbol{\omega}')}{(2M+1)^{p-h+1}}\right]^d\\ &= \sum_{\substack{k=1,\ldots,p\\ h=1,\ldots,k}}\beta^{p-h}\sum_{\substack{\boldsymbol{\omega}\in\Omega_{p,k}\\ \boldsymbol{\omega}'\in\Omega_{k,h}}}u(\boldsymbol{\omega}')v(\boldsymbol{\omega},\boldsymbol{\omega}')^d.\end{aligned} \quad (26)$$

The second equality in (26) holds since, for any given p, the sums $\sum_{\boldsymbol{\omega}\in\Omega_{p,k}}$ and $\sum_{\boldsymbol{\omega}'\in\Omega_{k,h}}$ are over a finite number of terms, and the coefficients $u(\boldsymbol{\omega}')$ are finite and do not depend on M. Therefore, the limit operator can be swapped with the summations. In (26) the term $v(\boldsymbol{\omega},\boldsymbol{\omega}')$ is defined as

$$v(\boldsymbol{\omega},\boldsymbol{\omega}') = \lim_{M\to\infty}\frac{\psi_M(\boldsymbol{\omega},\boldsymbol{\omega}')}{(2M+1)^{p-h+1}}. \tag{27}$$

We now consider three possible cases:

- if $h=1$, then $\Omega_{k,1}=\{[\underbrace{1,\ldots,1}_{k}]\}$, thus we only consider $\boldsymbol{\omega}'=[\underbrace{1,\ldots,1}_{k}]$. Then, $\mathcal{P}_1(\boldsymbol{\omega}')=\{1,\ldots,k\}$ and the argument of the $\delta_K(\cdot)$ function in (25) is given by

$$\begin{aligned}\sum_{i'\in\mathcal{P}_1(\boldsymbol{\omega}')}\hat{w}_{i'}(\boldsymbol{\omega}) &= \sum_{i'\in\{1,\ldots,k\}}\hat{w}_{i'}(\boldsymbol{\omega}) = \sum_{i'=1}^{k}\hat{w}_{i'}(\boldsymbol{\omega})\\ &= \sum_{i'=1}^{k}\sum_{i\in\mathcal{P}_{i'}(\boldsymbol{\omega})}\ell_i-\ell_{[i+1]}\\ &= \sum_{i=1}^{p}\ell_i-\ell_{[i+1]}=0\end{aligned}$$

  and by consequence $\delta_K\left(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}\hat{w}_{i'}(\boldsymbol{\omega})\right)=1$. Hence by passing to the limit in (27) we obtain

$$v(\boldsymbol{\omega},\boldsymbol{\omega}') = \int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\mathrm{d}\mathbf{y} \tag{28}$$

  where $\tilde{C}_j(\boldsymbol{\omega}) = C\left(-\mathrm{j}2\pi\beta^{1/d}w_j(\boldsymbol{\omega})\right)$, and in analogy with (20), we defined $w_j=\sum_{i\in\mathcal{P}_j(\boldsymbol{\omega})}y_i-y_{[i+1]}$, $y_i\in\mathbb{R}$, $i=1,\ldots,p$ and we denoted by $\mathbf{y}$ the vector $\mathbf{y}=[y_1,\ldots,y_p]^{\mathsf{T}}$;
- if $1<h<k$, the argument of the $\delta_K(\cdot)$ function in (25) is always a function of the indices ℓ_i. Thus $v(\boldsymbol{\omega},\boldsymbol{\omega}')$ is given by $\int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega}))\prod_{j'=1}^{h}\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')\mathrm{d}\mathbf{y}$ where $\tilde{D}_{j'}(\boldsymbol{\omega},\boldsymbol{\omega}')=\delta_D\left(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}w_{i'}(\boldsymbol{\omega})\right)$ and $\delta_D(\cdot)$ denotes the Dirac's delta;
- if $h=k$, the cardinality of $\Omega_{k,h}=\Omega_{k,k}$ is $S(k,k)=1$ and $\Omega_{k,k}=\{[1,\ldots,k]\}$. Thus, we only consider $\boldsymbol{\omega}'=[1,\ldots,k]$. It follows that:

$$\begin{aligned}&v(\boldsymbol{\omega},[1,\ldots,k])\\ &\quad=\int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\prod_{j'=1}^{k}\tilde{D}_{j'}(\boldsymbol{\omega},[1,\ldots,k])\mathrm{d}\mathbf{y}\\ &\quad=\int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\tilde{D}_j(\boldsymbol{\omega},[1,\ldots,k])\mathrm{d}\mathbf{y}.\end{aligned}$$

  Since $\mathcal{P}_j([1,\ldots,k])=\{j\}$ then $\tilde{D}_j(\boldsymbol{\omega},[1,\ldots,k])=\delta_D\left(w_j(\boldsymbol{\omega})\right)$. Moreover $C(0)=1$, then we have

$$\begin{aligned}v(\boldsymbol{\omega},[1,\ldots,k]) &= \int_{\mathcal{H}_p}\prod_{j=1}^{k}\tilde{C}_j(\boldsymbol{\omega})\delta_D\left(w_j(\boldsymbol{\omega})\right)\mathrm{d}\mathbf{y}\\ &= \int_{\mathcal{H}_p}\prod_{j=1}^{k}C(0)\delta_D\left(w_j(\boldsymbol{\omega})\right)\mathrm{d}\mathbf{y}\\ &= \int_{\mathcal{H}_p}\prod_{j=1}^{k}\delta_D\left(w_j(\boldsymbol{\omega})\right)\mathrm{d}\mathbf{y}.\end{aligned}$$

As a last remark, if $k=1$, we have $h=1$ and $\Omega_{p,k}=\Omega_{p,1}=\{[\underbrace{1,\ldots,1}_{p}]\}$. Then $w_j(\boldsymbol{\omega})=\sum_{i=1}^{p}w_i=0$. Using (28), we obtain $v(\boldsymbol{\omega},\boldsymbol{\omega}')=\int_{\mathcal{H}_p}\prod_{j=1}^{k}C(0)\mathrm{d}\mathbf{y}=1$

APPENDIX C
PROOF OF LEMMA B.1

Recall that $\mathcal{M}(\boldsymbol{\omega})$ denotes the set of vectors $\boldsymbol{\mu}=[\mu_1,\ldots,\mu_p]$ inducing a given partition $\boldsymbol{\omega}$. As defined in Section III-B, if $\boldsymbol{\omega}\in\Omega_{p,k}$, then each $\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})$ contains k distinct values, namely, $\boldsymbol{\gamma}=[\gamma_1,\ldots,\gamma_k]$ where $0\le\gamma_j<r$, $j=1,\ldots,k$ and $\gamma_j\ne\gamma_{j'}$ for each $j,j'=1,\ldots,k$ and $j\ne j'$. Therefore, from (B.1) we can write

$$\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j} = \sum_{\substack{\gamma_1,\ldots,\gamma_k\\ \ne}}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j}$$

where the symbol $\sum_{\substack{\gamma_1,\ldots,\gamma_k\\ \ne}}$ indicates a sum over the variables $\gamma_1,\ldots,\gamma_k$ with the constraint that $\gamma_j\ne\gamma_{j'}$ for every $j,j'=1,\ldots,k$ and $j\ne j'$. Notice that the values γ_j ($j=1,\ldots,k$) are the scalar counterparts of the integer vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$, $\mathbf{v}_j=[v_{j1},\ldots,v_{jd}]^{\mathsf{T}}$, $0\le v_{jm}<\rho$, $m=1,\ldots,d$, through the invertible function $\mu(\cdot)$, i.e., $\gamma_j=\mu(\mathbf{v}_j)$, $j=1,\ldots,k$. Hence, by definition of $\bar{\mathbf{x}}$, we have $\bar{\mathbf{x}}_{\gamma_j}=\bar{\mathbf{x}}_{\mu(\mathbf{v}_j)}=\mathbf{v}_j$ and

$$\begin{aligned}\sum_{\boldsymbol{\mu}\in\mathcal{M}(\boldsymbol{\omega})}\prod_{j=1}^{k}\zeta^{\bar{\mathbf{x}}_{\gamma_j}^{\mathsf{T}}\hat{\mathbf{w}}_j} &= \sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_k\\ \ne}}\prod_{j=1}^{k}\zeta^{\mathbf{v}_j^{\mathsf{T}}\hat{\mathbf{w}}_j}\\ &= \sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_k\\ \ne}}\zeta^{\mathbf{v}_1^{\mathsf{T}}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_k^{\mathsf{T}}\hat{\mathbf{w}}_k}.\end{aligned} \tag{29}$$

We now compute the last term of (29) by summing over one variable at a time. We first notice that, for every set $\mathbf{v}_1,\ldots,\mathbf{v}_n$ of distinct vectors

$$\sum_{\mathbf{v}\ne\mathbf{v}_1,\ldots,\mathbf{v}_n}\zeta^{\mathbf{v}^{\mathsf{T}}\hat{\mathbf{w}}} = \begin{cases} r-n & \hat{\mathbf{w}}=\mathbf{0}\\ -\sum_{j=1}^{n}\zeta^{\mathbf{v}_j^{\mathsf{T}}\hat{\mathbf{w}}} & \hat{\mathbf{w}}\ne\mathbf{0}.\end{cases}$$

In particular when $\mathbf{w}\ne\mathbf{0}$, $\sum_{\mathbf{v}}\zeta^{\mathbf{v}^{\mathsf{T}}\hat{\mathbf{w}}}=0$.

Let us arbitrarily choose the variable $\mathbf{v}_k$. If by hypothesis $\mathbf{w}_k\ne\mathbf{0}$, then by summing (29) over $\mathbf{v}_k$ we get

$$\begin{aligned}&\sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_k\\ \ne}}\zeta^{\mathbf{v}_1^{\mathsf{T}}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_k^{\mathsf{T}}\hat{\mathbf{w}}_k}\\ &\qquad= -\sum_{j=1}^{k-1}\sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\\ \ne}}\zeta^{\mathbf{v}_1^{\mathsf{T}}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_{k-1}^{\mathsf{T}}\hat{\mathbf{w}}_{k-1}}\zeta^{\mathbf{v}_j^{\mathsf{T}}\hat{\mathbf{w}}_k}.\end{aligned} \tag{30}$$

We compute separately each of the $k-1$ contributions in (30). In particular, the generic j'th term $(j=j')$ is given by

$$-\sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\\ \neq}} \zeta^{\mathbf{v}_1^\mathsf{T}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_{k-1}^\mathsf{T}\hat{\mathbf{w}}_{k-1}}\zeta^{\mathbf{v}_{j'}^\mathsf{T}\hat{\mathbf{w}}_k} = -\sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_{k-1}\\ \neq}} \zeta^{\mathbf{v}_1^\mathsf{T}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_{j'}^\mathsf{T}(\hat{\mathbf{w}}_{j'}+\hat{\mathbf{w}}_k)+\mathbf{v}_{k-1}^\mathsf{T}\hat{\mathbf{w}}_{k-1}}.$$

We now proceed by summing over the variable $\mathbf{v}_{j'}$. If by hypothesis $\hat{\mathbf{w}}_{j'}+\hat{\mathbf{w}}_k\neq\mathbf{0}$, this summation produces $k-2$ terms. Again, we consider each term separately. This procedure repeats until a subset $\mathcal{S}$ of $\{1,\ldots,k\}$ is found, such that $\mathbf{s}=\sum_{i\in\mathcal{S}}\hat{\mathbf{w}}_i=\mathbf{0}$.

In this case, the contribution of the nth sum is given by $r-(k-n)$ where $n=|\mathcal{S}|$ is the cardinality of $\mathcal{S}$. Overall, after n sums the total contribution is

$$(-1)^{n-1}(n-1)!(r-(k-n)) \times \sum_{\substack{\mathbf{v}_j, j\in\{1,\ldots,k\}-\mathcal{S}\\ \neq}} \prod_{j\in\{1,\ldots,k\}-\mathcal{S}} \zeta^{\mathbf{v}_j^\mathsf{T}\hat{\mathbf{w}}_j}.$$

The factor $(n-1)!$ accounts for the number of permutations of the elements in $\mathcal{S}$, once the first element is fixed (remember that we arbitrarily chose the first variable of the summation). The factor $(-1)^{n-1}$ takes into account that we summed $n-1$ times with the condition $\hat{\mathbf{w}}\neq\mathbf{0}$, which implies $n-1$ sign changes. Eventually, the term $\sum_{\substack{\mathbf{v}_j, j\in\{1,\ldots,k\}-\mathcal{S}\\ \neq}}\prod_{j\in\{1,\ldots,k\}-\mathcal{S}}\zeta^{\mathbf{v}_j^\mathsf{T}\hat{\mathbf{w}}_j}$ is similar to the last term in (29) where only $k-n$ variables $\mathbf{v}$ are involved.

This procedure repeats until we sum over all variables $\mathbf{v}$. This is equivalent to check if for all possible partitions of $\{1,\ldots,k\}$ in h subsets $\mathcal{P}_1,\ldots,\mathcal{P}_h$, $h=1,\ldots,k$ the condition $\mathbf{s}_1=\mathbf{s}_2=\cdots=\mathbf{s}_h=\mathbf{0}$ holds, with $\mathbf{s}_j=\sum_{i\in\mathcal{P}_j}\hat{\mathbf{w}}_i$, $n_j=|\mathcal{P}_j|$, and $\sum_j n_j=k$. In this case, the contribution is given by $\prod_{j=1}^{h}(-1)^{n_j-1}(n_j-1)!p_r(n_1,\ldots,n_h)$ and it is 0 otherwise. Here $p_r(n_1,\ldots,n_h)=(r-(k-n_1))(r-(k-n_1-n2))\cdots(r-(k-n_1-n_2-\cdots-n_{h-1}))$. In conclusion, we can write

$$\sum_{\substack{\mathbf{v}_1,\ldots,\mathbf{v}_k\\ \neq}} \zeta^{\mathbf{v}_1^\mathsf{T}\hat{\mathbf{w}}_1+\cdots+\mathbf{v}_k^\mathsf{T}\hat{\mathbf{w}}_k} = \sum_{h=1}^{k}\sum_{\boldsymbol{\omega}'\in\Omega_{k,h}} u(\boldsymbol{\omega}')p_r(\boldsymbol{\omega}')\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}')$$

where $\tilde{K}(\boldsymbol{\omega},\boldsymbol{\omega}')=\prod_{j'=1}^{h}\delta_K(\sum_{i'\in\mathcal{P}_{j'}(\boldsymbol{\omega}')}\hat{\mathbf{w}}_{i'}(\boldsymbol{\omega}))$ and $\delta_K(\cdot)$ is the Kronecker's delta function, $u(\boldsymbol{\omega}')=(-1)^{k-h}\prod_{j'=1}^{h}(|\mathcal{P}_{j'}(\boldsymbol{\omega}')|-1)!$ and $p_r(\boldsymbol{\omega}')$ is a polynomial in r of degree h. For large r, $p_r(\boldsymbol{\omega}')\simeq r^h$, thus, proving the lemma.

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