# Analog Beamforming With Low Resolution Phase Shifters

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***Abstract*—In this letter, we consider analog beamforming using low resolution phase shifters for millimeter waves communications. We propose a hierarchical codebook design, where the beamforming vectors in the codebook are grouped into multiple levels and the preferred beamforming vector at each level is constructed to approximate an amplitude beamforming gain mask by using a low complexity local search algorithm. We show, by simulations, that the proposed codebook using low resolution phase shifters outperforms the existing schemes using high resolution phase shifters.**

***Index Terms*—Analog beamforming, millimeter waves, phase shifters, hierarchical codebook.**

## I. Introduction

MILLIMETER waves (mm-Waves) offer a wide range of spectrum frequencies that make them a potential candidate for 5G wireless communication systems [1]. One major issue of mm-Wave communications is the extremely high path loss. This can be typically overcome by implementing *transmit and receive beamforming* through multiple antennas. In the literature, there are three types of beamforming techniques: digital beamforming [2], analog beamforming [3], [4], and hybrid beamforming [5], [6].

In this letter, we focus on analog beamforming only [3], [4], which uses one transmit/receive radio-frequency (RF) chain. Optimal *transmit* and *receive beamforming vectors* in analog beamforming are selected to align the beams along the strongest path of the channel. The best beamforming vectors are chosen to modify phases of analog domain signals. These phase modifications can be realized by using phase shifters [7] that operate over a small amount of distinct phase shifts, as a large number of phase shifts would require very high precision components that are hard to realize.

In analog beamforming, finding optimal beamforming vector requires full channel state information (CSI) at both transmitter and receiver, i.e., channel estimation is needed. Thanks to the sparseness of mm-Wave channels, the channel estimation is equivalent to finding the steering angles and path-loss coefficients for each of the different paths. In the special case of uniform linear array (ULA) antennas, the channel is fully described by the angle-of-departure (AoD) and angle-of-arrival (AoA) of each path.

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In the literature, there are various beamforming schemes for finding the best beamforming vectors (e.g., [3] and [8]). We consider the real-time analog beamforming adaptive to instantaneous CSI rather than the statistical CSI based method, since the former is suitable for the case considered in this letter: one RF chain per antenna, while the latter is for a large number of RF chains per antenna. To find the best beamforming vectors, an exhaustive search method was used in [6], but it requires a long training time in order to test all beamforming vector pairs. To overcome this problem, in [3] and [4], a hierarchical beam search was proposed, which divides the search into several levels, starting from a wide beam at first level and narrowing it at each successive level. The last level has the narrowest beam and highest beamforming gain. Thus the challenging task is to design a *codebook* of beamforming vectors for different beam-widths. In [3], a three-level hierarchical codebook was designed, but with a limited beamforming gain. Orthogonal matching pursuit based approach was used in [5] to find the codebook for hybrid beamforming schemes.

In this letter, we consider analog beamforming using low resolution phase shifters ($2, 3, 4$ bits), while [4] and [5] use full and high resolutions phase shifters, respectively. We propose a hierarchical codebook design for a Tx/Rx terminal with an arbitrary number of antennas. The preferred beamforming vectors are selected to approximate an *amplitude beamforming gain mask* (see (4)) using a low complexity local search algorithm (LSA), which searches among all $q^N$ possible choices for a terminal of $N$ antennas. In contrast, in [4], the beamforming vector has some zero elements and in [5], the beamforming vector is restricted to take a particular form. By simulations, we show our codebook with low resolution phase shifters outperforms existing schemes with high resolution phase shifters.

## II. System Model

We consider a point-to-point mm-Wave communication system, where the transmitter and receiver have $N_t$ and $N_r$ antennas, respectively, and each has only one RF chain. The phase shifters are assumed to operate on $q$ angles that are spaced uniformly in $[0, 2\pi)$. The set of all possible beamforming vectors for a terminal with $N$ antennas is denoted by

$$\mathcal{S}_q(N) = \Big\{\mathbf{w} \in \mathbb{C}^{N\times 1} : w_i = e^{j\beta_i}, \\ \beta_i \in \Big\{0, \frac{2\pi}{q}, \ldots, 2\pi\frac{q-1}{q}\Big\}\ \forall\ i = 0, 1, \ldots, N-1\Big\}, \quad (1)$$

where $\|\mathbf{w}\|^2 = N$ and $|\mathcal{S}_q(N)| = q^N$. We let $\mathbf{w}_t$ and $\mathbf{w}_r$ denote the transmit and receive beamforming vectors, chosen from codebooks $\mathcal{C}_t \subset \mathcal{S}_q(N_t)$ and $\mathcal{C}_r \subset \mathcal{S}_q(N_r)$, respectively.

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*Channel Model:* Let the antenna response vector in the angular direction $\theta$ be $\mathbf{a}(\theta, N) \triangleq [1, e^{j\pi\cos(\theta)}, \ldots, e^{j\pi(N-1)\cos(\theta)}]^T$ then the millimeter wave channel can be written as $\mathbf{H} = \sum_{l=1}^{L} \alpha_l \mathbf{a}(\phi_l, N_r)\mathbf{a}(\theta_l, N_t)^H$ where $\alpha_l \sim \mathcal{CN}(0, \sigma_{\alpha_l}^2)$ is the path-loss coefficient of the $l^{th}$ path such that $\sum_{l=1}^{L} \sigma_{\alpha_l}^2 = 1$, $\phi_l$ and $\theta_l$ are the corresponding AoA and AoD, respectively, and $L$ is the total number of paths. Since the path loss and materials absorption are high at mm-Wave frequencies, the number of paths can usually be $L = 3$ or 4. We assume a uniform linear array (ULA) at both transmitter and receiver with antenna spacing by half wavelength.

Let $x$ denote the baseband equivalent complex symbol sent through the transmitter phase shifters $\mathbf{w}_t \in \mathcal{C}_t \subset \mathcal{S}_q(N_t)$, then the received signal after receiver phase shifters $\mathbf{w}_r \in \mathcal{C}_r \subset \mathcal{S}_q(N_r)$ is

$$y = \frac{1}{\sqrt{N_r N_t}} \sum_{l=1}^{L} \alpha_l \mathbf{w}_r^H \mathbf{a}(\phi_l, N_r)\mathbf{a}(\theta_l, N_t)^H \mathbf{w}_t\, x + \mathbf{n}, \tag{2}$$

where $\frac{1}{\sqrt{N_r}}$ and $\frac{1}{\sqrt{N_t}}$ are the normalization factors for $\mathbf{w}_r$ and $\mathbf{w}_t$ respectively, and $\mathbf{n} = (n_1, \ldots, n_{N_r})^T$ is the received noise vector with i.i.d entries (i.e., $n_i \sim \mathcal{CN}(0, \sigma^2)$). Here $|\mathbf{w}_r^H \mathbf{a}(\phi_l, N_r)|$ and $|\mathbf{a}(\theta_l, N_t)^H \mathbf{w}_t|$ represent the beamforming gains of $\mathbf{w}_r$ and $\mathbf{w}_t$ in angular directions $\phi_l$ and $\theta_l$, respectively.

We define the transmit signal-to-noise ratio (SNR) as $\mathrm{SNR_{Tx}} \triangleq P_x/\sigma^2$, where $P_x = E\{|x|^2\}$ denotes the transmitted power of the symbol $x$ and $\sigma^2$ is the noise power. Similarly, the receive SNR is defined as $\mathrm{SNR_{Rx}} \triangleq \frac{P_x |\mathbf{w}_r^H \mathbf{H} \mathbf{w}_t|^2}{\sigma^2 N_r N_t}$. The spectral efficiency of the link in (2) is given by [5]

$$C = \log_2\left[1 + \mathrm{SNR_{Rx}}\right] \ \text{bits/s/Hz}, \tag{3}$$

which depends on the beamforming vectors $\mathbf{w}_t$ and $\mathbf{w}_r$.

*Beamforming Protocol:* Considering the hierarchical codebook based protocol (e.g., [5]), the beamforming vectors in the codebook are selected to generate beams with preferred beam-widths and directions. In particular, the beamforming vectors in the codebook are grouped into $m$ different levels. For each level $\ell = 1, \ldots, m$, there are $K^\ell$ disjoint beams of decreasing beam-width. The protocol selects the narrowest beamforming vectors aligned with the strongest channel path from the transmit/receive codebooks at level $m$ by an exchange of pilot tones. Both transmitter and receiver cooperatively scan all the beamforming vectors in the codebook at level 1 ($K \times K$ pilot tones) and select the best one at that level. Then both of them repeat the search over $K^2$ beamforming vector pairs at level 2, which cover the same angular region of the best pair at level 1. This process is repeated until the last level $m$. This protocol only uses $mK^2$ pilot tones, which improves over the exhaustive search protocol in [6] with $K^{2m}$ pilot tones.

## III. The Codebook Design Using LSA

In this section, we present our hierarchical codebook design for analog beamforming at a Tx/Rx terminal with arbitrary $N$ antennas using low resolution phase shifters ($q = 4, 8, 16$).

Let us consider the discrete time Fourier transform (DTFT) of a beamforming vector $\mathbf{w}$, using a continuous frequency variable $\omega \in (-1, 1]$, $\mathcal{W}(e^{j\pi\omega}) = \sum_{n=0}^{N-1} \mathbf{w}(n)e^{-jn\pi\omega}$. The DTFT describes the beamforming gain of $\mathbf{w}$ in all azimuth angles $\psi = \cos^{-1}(\omega) \in [0, \pi)$. Therefore, $|\mathcal{W}(e^{j\pi\omega})|$ represents the amplitude beamforming gain given by $\mathbf{w}$ along the angular directions $\psi$. In order to efficiently plot the radiation pattern of the beamformer, we consider the DTFT at $R$ discrete points $z_1, z_2, \ldots, z_R$, which are equally spaced in $(-1, 1]$, i.e., $z_i = (-1 + \frac{2i}{R})$, for $i = 1, 2, \ldots, R$. Note that this results in a non-uniform angular resolution $(\Delta\psi)$ in the variable $\psi$, as well as a smooth diagram when $R > N$ is sufficiently large.

The corresponding DFT is obtained by defining an $N \times R$ matrix $\mathbf{A}$ such that $\mathbf{A}^H\mathbf{w} = [\mathcal{W}(e^{j\pi z_1}), \ldots, \mathcal{W}(e^{j\pi z_R})]^T$. Then $|\mathbf{A}^H\mathbf{w}| \triangleq [|\mathcal{W}(e^{j\pi z_1})|, \ldots, |\mathcal{W}(e^{j\pi z_R})|]^T$ denotes the vector of amplitude beamforming gains in the angular directions $\psi_i = \pm\cos^{-1}(z_i)$, for $i = 1, 2, \ldots, R$. Since $z_i = (-1 + \frac{2i}{R})$, the $(R/2)^{th}$ row of $\mathbf{A}^H$ has all one entries and $\mathbf{A}^H$ can be related to the first $N$ columns of an $R \times R$ DFT matrix $\mathbf{F} = \{e^{-j2\pi nk/R}\}_{k,n=0}^{R-1}$ by swapping the block of the first $\frac{R}{2} - 1$ rows with the block of last $\frac{R}{2} + 1$ rows, i.e., the submatrix with the first $N$ columns of $\mathbf{F}$ is given by $\mathbf{F}_N = \mathbf{P}\mathbf{A}^H$, where $\mathbf{P} = \begin{pmatrix} \mathbf{0} & \mathbf{I}_{\frac{R}{2}+1} \\ \mathbf{I}_{\frac{R}{2}-1} & \mathbf{0} \end{pmatrix}$ and $\mathbf{I}_r$ is an $r \times r$ identity matrix and $\mathbf{0}$ is an all zero matrix.

*Amplitude Beamforming Gain Mask*: The amplitude beamforming gain mask for the beams in the hierarchical codebook should have a constant amplitude in the main lobe and zero everywhere else. We let $\mathbf{g}(\ell, i)$, an $R$ component vector, denote the mask for level $\ell = 1, \ldots, m = \log_K N$ and $i = 1, \ldots, K^\ell$. The $j$-th component of $\mathbf{g}(\ell, i)$, for $j = 1, \ldots, R$, is given by

$$g_j(\ell, i) \triangleq \begin{cases} c_\ell & \text{if } \frac{R(i-1)}{K^\ell} < j \leq \frac{Ri}{K^\ell} \\ 0 & \text{otherwise} \end{cases}. \tag{4}$$

where $c_\ell = \sqrt{NK^\ell}$ (see Lemma 1 in Appendix A). That is, an ideal steering vector should result in a beam with a constant amplitude $c_\ell$ in the covered angular region $[\cos^{-1}(-1 + \frac{2i}{K^\ell}), \cos^{-1}(-1 + \frac{2(i-1)}{K^\ell}))$ and zero in the other angular positions. For example, for $\ell = 1, i = 1$, and $K = 2$, we have

$$\mathbf{g}(1, 1) = [\underbrace{c_1, \ldots, c_1}_{R/2 \text{ values}}, \underbrace{0, \ldots, 0}_{R/2 \text{ values}}]^T$$

covering the angular region $[\pi/2, \pi)$.

*Remark:* In principle it is possible to consider levels $\ell > \log_K N$, where all the beams maintain the same width and gain of level $\ell = \log_K N$, but can be steered to higher resolution angles. In this case, we can still use the non-overlapping masks in (4) with the same $c_\ell = N$, for $\ell > \log_K N$. The actual $K^\ell$ beams will have a larger overlap with a main lobe peak at the mid-angle of the mask and a minimum at the edge of the mask. Our simulations have shown minor performance improvements when $\ell > \log_K N$ (not reported here due to space limitations).

*Hierarchical Codebook Design*: Given an arbitrary steering vector $\mathbf{x} \in \mathcal{S}_q(N)$, we let

$$\zeta_{(\ell,i)}(\mathbf{x}) \triangleq \| |\mathbf{A}^H\mathbf{x}| - \mathbf{g}(\ell, i) \|^2 \tag{5}$$

for $\ell = 1, \dots, m$, and $i = 1, \dots, K^{\ell}$, be the error between the amplitude beamforming gain of $\mathbf{x}$ relative to the amplitude beamforming gain mask $\mathbf{g}(\ell, i)$. Therefore the optimum steering vector $\mathbf{w}(\ell, i) \in \mathcal{S}_q(N)$ is given by

$$\mathbf{w}(\ell, i) \triangleq \arg\min_{\mathbf{x} \in \mathcal{S}_q(N)} \zeta_{(\ell,i)}(\mathbf{x}). \tag{6}$$

Then the best hierarchical codebook can be obtained by $\mathcal{C} \triangleq \{\mathbf{w}(\ell, i) | \ell = 1, \dots, m,\ i = 1, \dots, K^{\ell}\}$. The following proposition gives the conditions for which (6) can be solved only once for each level.

*Proposition 1:* If $K^{\ell}$ divides $q$, then $w_s(\ell, p) = w_s(\ell, 1) e^{jr2\pi\left(\frac{p-1}{K^{\ell}}\right)}$, where $w_s(\ell, p)$ is the $s^{th}$ element in beamforming vector $\mathbf{w}(\ell, p)$, which corresponds to the phase shift of the $s^{th}$ antenna.

*Proof:* See Appendix B. ■

*Local Search Algorithm (LSA)*: In the following we drop the index $(\ell, i)$ for simplicity. An exhaustive search to solve (6) has an exponential complexity $|\mathcal{S}_q(N)| = q^N$. For example, in a mm-Wave system with $q = 4$ and $N = 64$, we need to compute $4^{64} (\approx 10^{38.5})$ values. To reduce the complexity, we propose an LSA, which provides a sub-optimal solution to (6). The LSA starts with an initial value of $\mathbf{x}$, which can be chosen randomly or obtained by the compressed sensing method proposed in [5]. Then the algorithm computes (5) for all the vectors in the neighborhood of solution $\mathbf{x}$, defined as

$$\mathcal{N}_d(\mathbf{x}) \triangleq \{\mathbf{y} : \mathbf{y} \in \mathcal{S}_q(N) \text{ and } 0 < \|\mathbf{x} - \mathbf{y}\|_0 \le d\}$$

where $\|\mathbf{x} - \mathbf{y}\|_0$ denotes the number of non-zero values in $(\mathbf{x} - \mathbf{y})$. Hence, we can interpret $\mathcal{N}_d(\mathbf{x})$ as the set of all vectors in $\mathcal{S}_q(N)$ that differ in at most $d$ positions from $\mathbf{x}$. The size of the neighborhood is $|\mathcal{N}_d(\mathbf{x})| = \sum_{i=1}^{d} \binom{N}{i} (q-1)^i$.

If the best solution found in the neighborhood has smaller $\zeta(\mathbf{x})$ than the present solution, then $\mathbf{x}$ is updated. This process stops when the present solution yields a smaller error than all its neighbors. To improve the performance, we can run the algorithm $r_s$ times, each time starting with different initial vector $\mathbf{x}$, and then select the solution with the least error. A similar approach applied to constant envelop multiuser precoding is proposed in [9].

*Complexity and convergence:* The complexity of the algorithm is $O(r_s |\mathcal{N}_d(\mathbf{x})|)$. Fig. 1 shows the variation of $\zeta(\mathbf{w})/R$ as a function of $d$ and $r_s$ for $N = 32$. We observed that the optimum metric is converging when $d = 2$ and $r_s = 1000$. Note that the codebook design is performed offline and hence large $r_s$ can be used.

In summary, the proposed LSA provides a heuristic solution for any given value of $K$ and $N$ with low resolution phase shifters. In contrast, in [4], the hierarchical codebooks using the deactivation (DEACT) method and beam widening via single RF sub-array (BMW-SS) method were designed for *non-quantized phase shifters* ($q \to \infty$) supporting a continuous range of phase shifts, which can only be implemented by high resolution phase shifters. Moreover, for some hierarchical levels in these codebooks, some antennas are turned off. Hence, to preserve a constant total power for all beams, the active antennas have a higher peak power requirement. Further, the BMW-SS approach was designed only for $N = K^p$, for

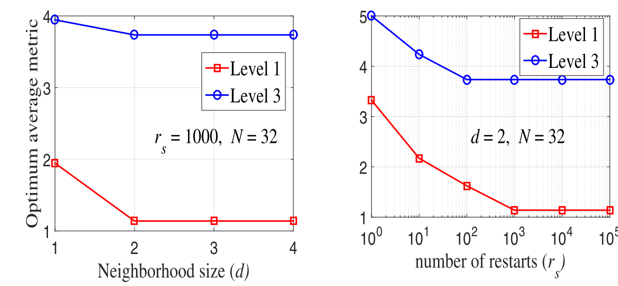


Fig. 1. The variation of optimum average metric with $d$ and $r_s$ for $N = 32$.

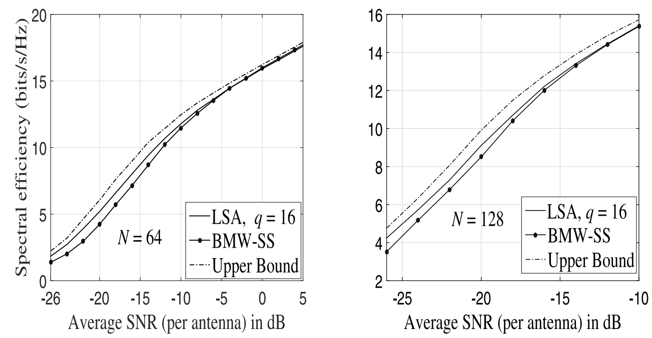


Fig. 2. The spectral efficiency of the proposed design codebook for different values of $N$ (64, 128) with $L = 3$, $q = 16$, $K = 2$, $r_s = 1000$, and $d = 2$.

some positive integer $p$, since it needs to divide the antennas into $K$ smaller sub-arrays, while our method uses all antennas to form the beams, which reduces the peak power.

Furthermore, the method in [5] finds the optimal solution and then quantizes it to the constrained beamforming vector in $\mathcal{S}_q(N)$, where the quantization requires high resolution phase shifters to reduce the overall error. In contrast, our method directly selects the best beamforming vector from the set $\mathcal{S}_q(N)$.

## IV. Simulation Results

In this section, we compare the spectral efficiencies of our hierarchical codebook for low resolution phase shifters ($q = 4, 8, 16$) and the other codebook using BMW-SS [4] for high resolution phase shifters ($q \to \infty$). In all simulations, we consider the per-antenna transmission power model in [4] and assume that the power per antenna is the same in all cases rather than a constant total power. We adopt the following parameters: $K = 2$, $N = 32,\ 64,\ 128$, $m = \log_2(N)$, $r_s = 1000$, and $d = 2$, and $L = 3$ (channel paths containing one line-of-sight (LoS) path and two non-line-of-sight (NLoS) paths). We assume the variance of the LoS path ($\eta$) is greater than that of the NLoS paths by 10 dB.

Fig. 2 illustrates the spectral efficiencies of our codebook with $q = 16$ and the BMW-SS one using high resolution phase shifters [4], for $N = 64, 128$, respectively. The *upper bound* is obtained by assuming the genie-aided receiver that knows perfect CSI and uses the *amplitude beamforming gain mask* in (4) for the beam search protocol. The perfect CSI case is plotted with the best beamforming vectors selected directly using full CSI. We observe that the performance of our codebook with low resolution phase shifters outperforms the BMW-SS

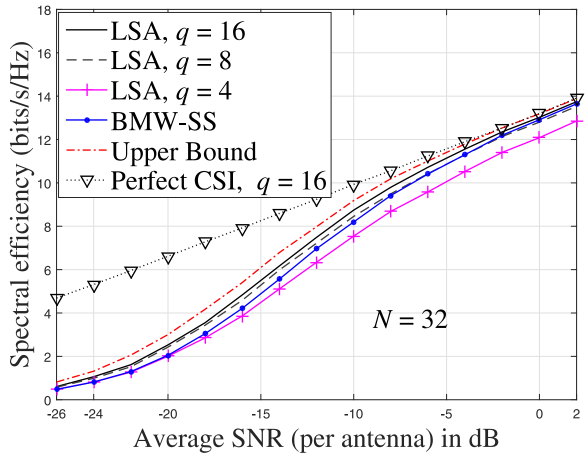


Fig. 3. The comparison of the proposed LSA design codebook with the BMW-SS method for different values of $q$ (4, 8, 16).

codebook with high resolution phase shifters. Similar observations can be found in Fig. 3 when $N = 32$, and $q = 8,\ 16$. When $q = 4$, the codebook using LSA method has similar performance to BMW-SS at low SNR region, but degrades in high SNR region. We also observe that the LSA performance is approaching the perfect CSI case at high SNR's for $q = 16$. Similar performance can also be found for the cases $q = 4$ and $q = 8$. We also compared the gain patterns and the search performance in terms of *success rate* (defined in [4]) between LSA and BMW-SS methods and verified that they have similar performance. Due to space limitation, we do not include these simulation results. Similar to the extension of the BMW-SS to hybrid beamforming in [10], our future work will consider the extension of our approach to the hybrid beamforming case.

## V. Conclusion

In this letter, we have proposed a hierarchical codebook design for analog beamforming with low resolution phase shifters. The beamforming vectors in our codebook are grouped into multiple levels. At each level, the preferred beamforming vector is constructed to approach the corresponding amplitude beamforming gain mask using a low complexity local search algorithm. Through simulations, we have shown that our codebooks with low resolution phase shifters outperform an existing scheme with high resolution phase shifters. Our design method can be extended to a variety of antenna arrangements, such as uniform planar arrays.

## Appendix A

*Lemma 1:* The value of $c_\ell$ in (4) is upper bounded by $\sqrt{NK^\ell}$.

*Proof:* The spectral norm of $\mathbf{A}^H$ is defined as $\|\mathbf{A}^H\|_2 = \max_{\|\mathbf{x}\|=1} \|\mathbf{A}^H\mathbf{x}\| = \lambda_{max}$, where $\lambda_{max}$ is the largest singular value of $\mathbf{A}^H$. Since $\mathbf{F}_N = \mathbf{P}\mathbf{A}^H$ and $\mathbf{P}^{-1} = \mathbf{P}^H$, we have, $\mathbf{A}\mathbf{A}^H = \mathbf{F}_N^H(\mathbf{P}^{-1})^H\mathbf{P}^{-1}\mathbf{F}_N = \mathbf{F}_N^H\mathbf{F}_N = R\ \mathbf{I}_N$ and $\lambda_{max} = \sqrt{R}$. Let us consider the value of $\|\mathbf{g}(\ell, i)\| = \|\mathbf{A}^H\mathbf{w}(\ell, i)\| = c_\ell\sqrt{R/K^\ell}$. Therefore, according to the spectral norm definition, $c_\ell\sqrt{R/K^\ell} \le \sqrt{N}\lambda_{max} = \sqrt{NR}$. Hence we obtain $c_\ell \le \sqrt{NK^\ell}$. ∎

## Appendix B
## Proof of Proposition 1

Let $\mathbf{f}(\ell, i) = \mathbf{A}^H\mathbf{w}(\ell, i)$ represent the gain pattern of the $i^{th}$ beamforming vector at the $\ell^{th}$ level for $1 \le i \le K^\ell$. The vector $\mathbf{f}(\ell, p)$, for $2 \le p \le K^\ell$, is simply the right circular shifted version of $\mathbf{f}(\ell, 1)$ with a shift of $\frac{R(p-1)}{K^\ell}$. That is, for $2 \le p \le K^\ell$,

$$f_n(\ell, 1) = \begin{cases} f_{n+\frac{R(p-1)}{K^\ell}}(\ell, p) & \text{if } 1 \le n \le R - \frac{R(p-1)}{K^\ell} \\ f_{n+\frac{R(p-1)}{K^\ell}-R}(\ell, p) & \text{if } R - \frac{R(p-1)}{K^\ell} + 1 \le n \le R \end{cases} \tag{7}$$

where $f_n(\ell, i)$, for $1 \le n \le R$, denotes the $n^{th}$ element in the vector $\mathbf{f}(\ell, i)$.

For $1 \le n \le R - \frac{R(j-1)}{K^\ell}$, $2 \le p \le K^\ell$, we have

$$f_{n+\frac{R(p-1)}{K^\ell}}(\ell, p) = \sum_{r=0}^{N-1} e^{-jr\pi(-1+\frac{2n}{R})} e^{-jr2\pi\left(\frac{p-1}{K^\ell}\right)} w_s(\ell, p). \tag{8}$$

Similarly, for $R - \frac{R(p-1)}{K^\ell} + 1 \le n \le R$, $2 \le p \le K^\ell$, we have

$$f_{n+\frac{R(p-1)}{K^\ell}-R}(\ell, p) = \sum_{r=0}^{N-1} e^{-jr\pi(-1+\frac{2n}{R})} e^{jr2\pi} e^{-jr2\pi\left(\frac{p-1}{K^\ell}\right)} w_s(\ell, p).$$

Finally, we obtain $f_n(\ell, 1) = \sum_{r=0}^{N-1} e^{-jr\pi(-1+\frac{2n}{R})} w_s(\ell, 1)$. Therefore, the condition in (7) is satisfied for

$$w_s(\ell, p) = w_s(\ell, 1)e^{jr2\pi\left(\frac{p-1}{K^\ell}\right)}. \tag{9}$$

Assuming $\mathbf{w}(\ell, 1) \in \mathcal{S}_q(N)$, then the beamforming vector $\mathbf{w}(\ell, p)$ found from the above equation will be in $\mathcal{S}_q(N)$ if the phase shift $r\left(\frac{p-1}{K^\ell}\right)$ falls in the allowed $q$ uniformly spaced angles in $[0, 2\pi)$. This condition is satisfied if $K^\ell$ divides $q$ as $r$ and $p$ are integers.

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