Integer Space-Time Block Codes for Practical MIMO Systems

J. Harshan and Emanuele Viterbo

Abstract—Full-rate space-time block codes (STBCs) achieve high spectral-efficiency by transmitting linear combinations of information symbols through every transmit antenna. However, the coefficients used for the linear combinations, if not chosen carefully, result in (*i*) large number of processor bits for the encoder and (*ii*) high peak-to-average power ratio (PAPR) values. In this letter, we propose a new class of full-rate STBCs called Integer STBCs (ICs) for multiple-input multiple-output (MIMO) fading channels. A unique property of ICs is the presence of integer coefficients in the code structure which enables reduced numbers of processor bits for the encoder and lower PAPR values. We show that the reduction in the number of processor bits is significant for small MIMO channels, while the reduction in the PAPR is significant for large MIMO channels. We also highlight the advantages of the proposed codes in comparison with the well known full-rate algebraic STBCs.

Index Terms—MIMO, integer codes, PAPR, quantization, full-rate, STBCs.

I. INTRODUCTION AND PRELIMINARIES

SPACE-TIME coding is a powerful technique for communication over multiple-input multiple-output (MIMO) fading channels. It is effective not only to combat the degrading effects of multi-path fading [1] but also to provide increased data rate [2]. For an $n \times n$ MIMO channel (with n transmit antennas and n receive antennas), a space-time block code (STBC), if appropriately designed, is known to provide a rate of n complex symbols per channel use [2], and a spatial diversity order of n^2 [1]. Such STBCs are referred to as full-rate, full-diversity STBCs.

In this paper, we address the implementation aspects of full-rate, full-diversity STBCs with emphasis on the following two metrics: (*i*) the number of bits for quantization [3] at the transmitter, which impacts the overall memory size, and (*ii*) the peak-to-average-power ratio (PAPR) [4], which influences the total power consumption [5]. We now explain the importance of incorporating quantization effects in the code design, and then address the PAPR issues. Consider the well known rate-1 Alamouti design [6] given by,

$$\mathbf{X}_A = \left[\begin{array}{cc} x_1 & -x_2^* \\ x_2 & x_1^* \end{array} \right], \tag{1}$$

where the variables x_1, x_2 take values from an underlying complex constellation $\mathcal{S}$ (e.g. QAM). For this case, encoding

an STBC codeword is straightforward, as the input bits (either uncoded or coded bits) are mapped to the symbols of $\mathcal{S}$, and the corresponding symbols are transmitted in the format of (1). Now, consider the rate-2 Golden code [7] design given by,

$$\mathbf{X}_G = \left[\begin{array}{cc} f_1(x_1, x_2) & f_2(x_3, x_4) \\ f_3(x_3, x_4) & f_4(x_1, x_2) \end{array} \right], \tag{2}$$

where $x_1, x_2, x_3, x_4 \in \mathcal{S}$ carry information symbols, and $\{f_j(\cdot)\}$ are some *irrational* linear combinations of these symbols [7]. For every codeword transmission, the encoding operation for the Golden code involves computation of linear combinations of information symbols with irrational coefficients. An alternate technique is to store all the codewords of the code, and use the repository as a look-up-table (LUT) to map the information bits to the codewords. However, this method has exponential memory requirements as the constellation size increases. When the Golden code is implemented with limited hardware, it has been shown in [8] that there will be substantial degradation in the error-performance due to the finite-precision encoding operations. Apart from the Golden code, all other well known full-rate algebraic STBCs [9]-[17] have irrational coefficients in the code structure, which leads to degradation in the performance (with limited hardware).

In addition to the above problem, transmitting linear combinations of information symbols also increases the PAPR values if the coefficients are not carefully chosen. For linear power amplifiers, a high dynamic range in the transmitter hardware is expensive and consumes more power. On the other hand, non-linear amplifiers result in spectral spread leading to out-of-band radiation and in-band distortion [5]. Though the PAPR problem is salient in orthogonal frequency division multiplexing literature (where the information symbols are linearly combined through an inverse fast Fourier transform matrix), lower PAPR is always desirable even for single-carrier systems which transmit linear combinations of information symbols. Importantly, such PAPR disadvantages are severe for large MIMO channels as the number of information symbols to be linearly combined increases linearly with the number of transmit antennas.

The above-mentioned problems have motivated us to design a new class of full-rate STBCs with few bits for encoding operations and low PAPR values. For some prior works on the effects of quantization at the receiver for fading channels, we refer the reader to [18]-[20] and the references within. Henceforth, we only consider transmitter side quantization effects and assume that the receiver works with infinite precision. This work stems from the preliminary observations made in [8] and generalizes the code construction in [21] to any $n \times n$ MIMO channel.

$$\mathbf{X}_I = \begin{bmatrix} \langle \Phi_1, \mathbf{x}_1 \rangle & \langle \Phi_1, \mathbf{x}_2 \rangle & \langle \Phi_1, \mathbf{x}_3 \rangle & \cdots & \langle \Phi_1, \mathbf{x}_{n-1} \rangle & \langle \Phi_1, \mathbf{x}_n \rangle \\ \gamma \langle \Phi_2, \mathbf{x}_n \rangle & \langle \Phi_2, \mathbf{x}_1 \rangle & \langle \Phi_2, \mathbf{x}_2 \rangle & \cdots & \langle \Phi_2, \mathbf{x}_{n-2} \rangle & \langle \Phi_2, \mathbf{x}_{n-1} \rangle \\ \vdots & & & \vdots & \vdots & \vdots \\ \gamma \langle \Phi_{n-1}, \mathbf{x}_3 \rangle & \gamma \langle \Phi_{n-1}, \mathbf{x}_4 \rangle & \gamma \langle \Phi_{n-1}, \mathbf{x}_5 \rangle & \cdots & \langle \Phi_{n-1}, \mathbf{x}_1 \rangle & \langle \Phi_{n-1}, \mathbf{x}_2 \rangle \\ \gamma \langle \Phi_n, \mathbf{x}_2 \rangle & \gamma \langle \Phi_n, \mathbf{x}_3 \rangle & \gamma \langle \Phi_n, \mathbf{x}_4 \rangle & \cdots & \gamma \langle \Phi_n, \mathbf{x}_n \rangle & \langle \Phi_n, \mathbf{x}_1 \rangle \end{bmatrix} \tag{3}$$

II. SIGNAL MODEL

The $n \times n$ MIMO channel, denoted by the matrix $\mathbf{H}$, consists of a source and a destination terminal each equipped with n antennas. For $i, j \in \{1, 2, \ldots, n\}$, the channel between the ith transmit antenna and the jth receive antenna is assumed to be flat fading and hence, denoted by $[\mathbf{H}]_{i,j}$. Each $[\mathbf{H}]_{i,j}$ remains constant for a block of T ($T \geq n$) complex channel uses and is assumed to take an independent realization in the next block. Statistically, we assume $[\mathbf{H}]_{i,j} \sim \mathcal{N}_c(0, 1) \, \forall i, j$ across quasi-static intervals. We assume a coherent MIMO channel where only the receiver has the complete knowledge of $\mathbf{H}$. The source conveys information to the destination through an $n \times n$ full-rate STBC denoted by $\mathcal{C}$. We assume that a linear design $\mathbf{X}_{\mathcal{LD}}$ in complex variables $x_{r,s}$ for $r, s \in \{1, 2, \ldots, n\}$ (a total of n^2 symbols) is used to generate $\mathcal{C}$ by taking values from an underlying complex constellation $\mathcal{S}$. If $\mathbf{X} \in \mathcal{C}$ denotes a transmitted codeword matrix such that $E[|[\mathbf{X}]_{i,j}|^2] = P_s \, \forall i, j$, then the received matrix $\mathbf{Y} \in \mathbb{C}^{n \times n}$ at the destination is given by,

$$\mathbf{Y} = \sqrt{\frac{1}{n}} \mathbf{H} \mathbf{X} + \mathbf{Z},$$

where $\mathbf{H} \in \mathbb{C}^{n \times n}$ denotes the channel matrix, $\mathbf{Z} \in \mathbb{C}^{n \times n}$ denotes the additive white Gaussian noise (AWGN) with its entries distributed as $\mathcal{N}_c(0, \sigma^2)$. With this, the average receive signal power-to-noise ratio (SNR) per receive antenna is P_s/σ^2. If

$$\eta = \max |[\mathbf{X}]_{i,j}|^2 / E[|[\mathbf{X}]_{i,j}|^2]$$

denotes the PAPR [4], then we define the peak signal power-to-noise ratio (denoted by PSNR) as $\eta P_s/\sigma^2$.

We represent the parameters of the design $\mathbf{X}_{\mathcal{LD}}$ and the symbols of $\mathcal{S}$ using $(q-1)$-bits for the fractional part and 1-bit for the sign part. For any $y \in \mathbb{R}$ such that $|y| \leq 1$, we define the q-bit quantized version of y as $Q_q(y) \triangleq \lfloor y 2^{q-1} \rceil / 2^{q-1}$, where $\lfloor x \rceil$ represents the nearest integer to x. To avoid quantization errors due to *over-flows*, we scale down all the parameters using a suitable scale factor such that each component falls within the quantization interval $[-1, 1)$. With the above fixed-point representation, the q-bit encoding operation is only subject to quantization errors due to rounding operations in the least significant bits.

III. INTEGER STBCs

We now propose full-rate STBCs which need reduced number of bits for encoding operations with M-QAM. These codes do not contain irrational coefficients, and hence, can be exactly represented in finite-precision. We refer to such codes as Integer STBCs (ICs). These codes can be obtained by the linear design given in (3) (given at the top of this page) where

for $j, k \in \{1, 2, \ldots, n\}$, $\langle \Phi_k, \mathbf{x}_j \rangle$ denotes the inner product of the kth row of the circulant matrix

$$\Phi \triangleq \begin{bmatrix} 1 & \alpha & \alpha^2 & \cdots & \alpha^{n-2} & \alpha^{n-1} \\ \alpha^{n-1} & 1 & \alpha & \cdots & \alpha^{n-3} & \alpha^{n-2} \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \alpha^2 & \alpha^3 & \alpha^4 & \cdots & 1 & \alpha \\ \alpha & \alpha^2 & \alpha^3 & \cdots & \alpha^{n-1} & 1 \end{bmatrix},$$

and the vector $\mathbf{x}_j = [x_{j,1} \; x_{j,2} \; x_{j,3} \; \ldots \; x_{j,n}]$. The variables $x_{j,1}, x_{j,2}, \ldots, x_{j,n}$ carry information symbols from a square M-QAM given by

$$\mathcal{S} = \left\{ a + b\imath \mid a, b \in \{-2^{\frac{m}{2}} + 1, \ldots, -1, 1, \ldots, 2^{\frac{m}{2}} - 1\} \right\},$$

where $M = 2^m$ and m is even. We choose Φ to be a circulant matrix in order to improve the trace metric of the STBCs, and this point will be discussed in Section III-C. For the linear design in (3), the parameters γ and α have to be appropriately chosen to optimize the performance of the code. In particular, we have used the notation γ in order to help the reader to form connections to algebraic STBCs [14], [15]. For the 2^m-QAM constellation, we choose $\gamma = \imath$ (where $\imath = \sqrt{-1}$) and $\alpha = 2^{\frac{m}{2}}$ in the linear design. Note that $\alpha = 2^{\frac{m}{2}}$ is the smallest positive integer that results in distinct M^n complex points for $\langle \Phi_k, \mathbf{x}_j \rangle$ when using M-QAM. Further, any integer value for α larger than $2^{\frac{m}{2}}$ will consume more power, while any non-integer value for α requires higher quantization bits. Also, it is worth noting that phase shift keying (PSK) constellations are not used as the underlying complex symbols require higher number of bits for quantization.

Example 1: For 2×2 MIMO, an Integer STBC can be obtained using the design,

$$\begin{bmatrix} x_{1,1} + \alpha x_{1,2} & x_{2,1} + \alpha x_{2,2} \\ \gamma(\alpha x_{2,1} + x_{2,2}) & \alpha x_{1,1} + x_{1,2} \end{bmatrix}. \tag{4}$$

For the above design, the parameter α takes the value of 2, 4, and 8 when the variables $x_{1,1}, x_{1,2}, \ldots, x_{2,2}$ take values from 4-, 16-, and 64-QAM constellations, respectively.

It is important to note that the design in (3) is linear, and hence, ICs are sphere-decodable [21]. In particular, the 2×2 ICs have been shown to exhibit reduced sphere decoding complexity [21]. In general, we observe that the advantages in the complexity reduction diminish with larger dimensions.

Remark 1: Note that the proposed design in (3) resembles those of other algebraic STBCs [14], [15]. However, in this case, such an algebraic structure fails over the finite ring $\mathbb{Z}_{2^{m/2}}$ for $m \geq 1$.

A. Reduced Number of Processor Bits for ICs

Since the coefficients of the design and the entries of $\mathcal{S}$ belong to a finite subset of $\mathbb{Z}[i]$, the fixed-point encoding

TABLE I
PAPR (IN DB SCALE) FOR INTEGER CODES AND PERFECT CODES (GC-GOLDEN CODE, PC- PERFECT CODE). FOR $n = 3$, HEX CONSTELLATIONS ARE USED FOR THE PERFECT CODES

| MIMO dimension	STBCs	4-point constellation	16-point constellation	64-point constellation
$n = 2$	GC	2.77	5.32	6.45
	IC	2.55	4.21	4.62
$n = 3$	PC	4.31	6.86	7.99
	IC	3.67	4.63	4.75
$n = 4$	PC	5.74	8.29	9.42
	IC	4.22	4.73	5.59

Fig. 1. Normalized trace metric for Integer codes for several transmit antennas and QAM sizes (4-, 16-, 64-, and 256-QAM).

operations can be performed with reduced number of bits. Further, as $\mathcal{S}$ is a square 2^m-QAM and $\alpha = 2^{\frac{m}{2}}$, the in-phase and quadrature components of the complex numbers of the matrices are in the interval $[-d, d]$, where $d = 2^{\frac{mn}{2}} - 1$. Hence, the proposed ICs can be encoded using $\frac{mn}{2} + 1$ bits per dimension with a 2^m-QAM constellation. For the special case of 2×2 MIMO, the proposed ICs can be encoded using 3, 5, and 7 bits for 4-, 16-, and 64-QAM constellations, respectively. Note that these numbers are smaller when compared with that of the Golden code (7, 8, and 9 bits for 4-, 16-, and 64-QAM, respectively) and the Silver code (6, 9, and 9 bits for 4-, 16-, and 64-QAM, respectively) [21]. For larger MIMO channels (i.e., with large values of n), the dynamic range for the complex numbers increases, which in turn requires higher number of processor bits for the encoder. Thus, ICs provide significant reduction in the number of processor bits in small MIMO channels.

B. Low PAPR of ICs

We use the PAPR definition given in Section II. For ICs since $\alpha = 2^{\frac{m}{2}}$ and $\gamma = \iota$, the complex symbols transmitted through each antenna take values from a regular M^n-QAM constellation. As a result, the PAPR of the ICs is same as the PAPR of M^n-QAM. For perfect codes [14], it is well known that the transmitted complex numbers take values from a *rotated* and *unequally spaced* 2-dimensional constellation, which in turn results in larger value of PAPR when compared with that of the regular M^n-QAM. In Table I, we list the PAPR values of several full-rate codes when the information symbols come from 4-, 16-, and 64-QAM constellations. For the values corresponding to $n = 3$, HEX constellations are used for the Perfect code [14], while for $n = 2$ and 4, regular QAM is used (for $n = 3$, Perfect codes which are optimized for HEX constellations are explicitly available in [14], and hence, we have used them in the numerical results). The numbers in Table I highlight the PAPR advantages of the ICs especially for larger MIMO channels and larger QAM constellations.

C. Trace and Determinant Properties of ICs

In the literature on STBC design, the minimum determinant criterion [1] is used for high SNR error performance, whereas the minimum trace criterion [6] is used for the low and moderate SNR error performance. For the ICs, we have adhered to the trace criterion by using the circulant matrix Φ, which spreads the weight of α over different information symbols in every layer. For the specific case of 2×2 MIMO,
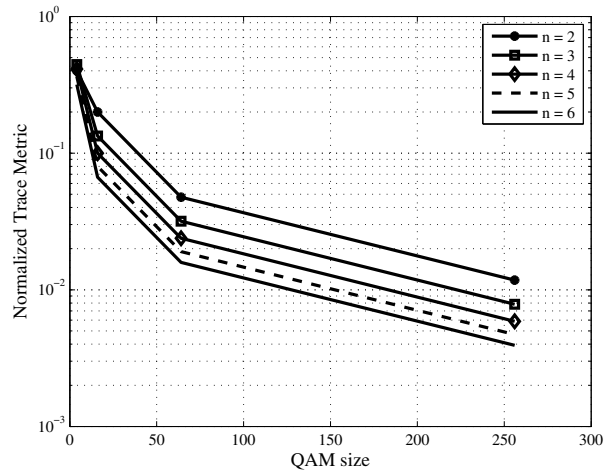
the position of α is swapped in the design from $x_{1,1} + \alpha x_{1,2}$ to $\alpha x_{1,1} + x_{1,2}$. A similar technique is adopted for the off-diagonal elements where the position of α is swapped from $x_{2,1} + \alpha x_{2,2}$ to $\alpha x_{2,1} + x_{2,2}$. As shown in [21], this technique results in a high value of the minimum trace metric. For larger MIMO channels, Fig. 1 shows the minimum trace values which are normalized by the average transmit power (per channel use) of ICs. We point out that the ICs do not satisfy the minimum determinant criterion. However, the number of codeword pairs that do not satisfy the determinant criterion is very small. For the 2×2 MIMO, the percentage of such pairs is reported in [21] for several QAM constellations. We note that ICs do not (and are not designed to) achieve the well known diversity multiplexing trade-off property. Instead, ICs are practical codes that provide good results at low and moderate SNR values due to (*i*) a high value of the minimum trace metric, and (*ii*) a very low percentage of the zero determinant codeword pairs.

D. Simulation Results on the Error Performance of ICs

In Fig. 2, we compare the codeword error rate (CER) of 2×2 ICs with the infinite-precision versions of the Golden code. We use PSNR (as defined in Section II) to capture the PAPR differences among the codes under comparison. If we compare the ICs with the finite-precision versions of the Golden codes (with 3, 5, and 7 encoder bits for 4-QAM, 16-QAM, and 64-QAM, respectively), then ICs will outperform the Golden codes since the latter codes are known to experience error-floors at high SNR values [8]. Along the similar lines, in Fig. 3, we compare the CER of 4×4 ICs with the infinite-precision and finite-precision versions of the Perfect codes with 4 and 16-QAM. For the 4×4 ICs, we observe that the quantization advantage is prominent with 4-QAM, while the PAPR advantage is prominent for 16-QAM. In conclusion, ICs, in addition to requiring fewer bits for the encoder, keep the error performance quite close to that of the infinite-precision version of the perfect codes in the *practical* SNR range of interest.
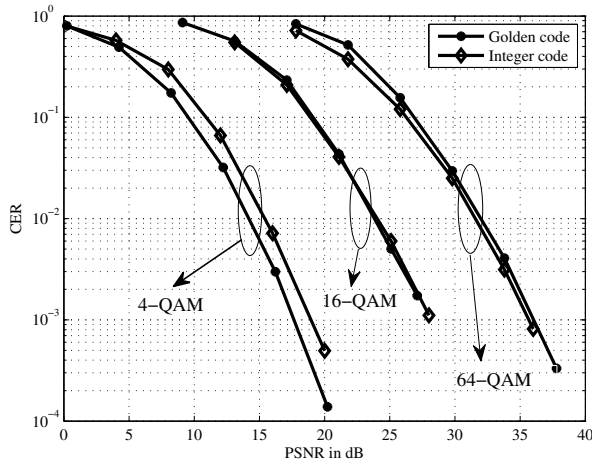
Fig. 2.　CER comparison of unquantized Golden code and Integer code for 2×2 MIMO with various QAM constellations.
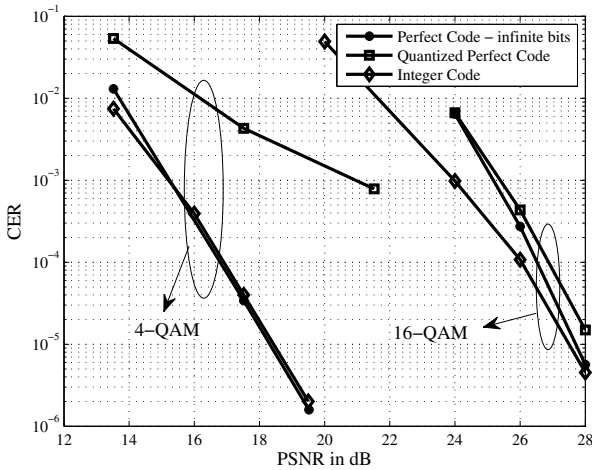

Fig. 3.　CER comparison of quantized Perfect code, unquantized Perfect code, and the Integer code for 4×4 MIMO with 4-QAM and 16-QAM. For quantized Perfect code and the Integer code, 5 bits and 9 bits are used for the encoder with 4-QAM and 16-QAM, respectively.

IV. SUMMARY AND DIRECTIONS FOR FUTURE WORK

We have proposed full-rate integer STBCs with reduced number of bits at the encoder and low PAPR values. We have shown that the reduction in the number of processor bits is significant for small MIMO channels, while the reduction in the PAPR is significant for large MIMO channels. An interesting direction for future work is to obtain an algebraic construction of full-rate integer STBCs over the ring $\mathbb{Z}_{2^{m/2}}$ for $m \geq 1$. Another direction for future work is to design ICs that are robust to quantization effects at the receiver.

REFERENCES

[1] V. Tarokh, N. Seshadri, and A. R. Calderbank, "Space-time codes for high data rate wireless communication: performance criterion and code construction," *IEEE Trans. Inf. Theory*, vol. 44, pp. 744–765, Mar. 1998.

[2] E. Telatar, "Capacity of multi-antenna Gaussian channels," *Europ. Trans. Telecommun.*, vol. 10, pp. 585–595, 1999.

[3] R. H. Walden, "Analog-to-digital converter survey and analysis," *IEEE J. Sel. Areas Commun.*, vol. 17, no. 4, pp. 539-550, Apr. 1999.

[4] Andrea Goldsmith, *Wireless Communications*. Cambridge University Press, 2005.

[5] B. Krongold and D. L. Jones, "PAR reduction in OFDM via active constellation extension," *IEEE Trans. Broadcast.*, vol. 49, no. 3, pp. 258–268, Sept. 2003.

[6] B. Vucetic and J. Yuan, *Space-Time Coding*. John Wiley and Sons, 2003.

[7] J. C. Belfiore, G. Rekaya, and E. Viterbo, "The Golden code: a 2×2 full-rate space-time code with non-vanishing determinants," *IEEE Trans. Inf. Theory*, vol. 51, no. 04, pp. 1432–1436, Apr. 2005.

[8] J. Harshan and E. Viterbo, "On the robustness of algebraic STBCs to coefficient quantization," in *Proc. 2012 IEEE Australian Communications Theory Workshop*, pp. 55–60.

[9] M. O. Damen, A. Tewfik, and J. C. Belfiore, "A construction of a space time code based on number theory," *IEEE Trans. Inf. Theory*, vol. 48, no. 3, pp. 753–760, Mar. 2002.

[10] C. Hollanti, J. Lahtonen, K. Ranto, R. Vehkalahti, and E. Viterbo, "On the algebraic structure of the silver Code: a 2×2 perfect space-time block code," in *Proc. 2008 IEEE Information Theory Workshop*, pp. 91–94.

[11] K. P. Srinath and B. S. Rajan, "Low ML-decoding complexity, large coding gain, full-rate, full-diversity STBCs for 2×2 and 4×2 MIMO systems," *IEEE J. Sel. Topics Signal Process.*, vol. 3, no. 6, pp. 916–927, Dec. 2009.

[12] H. Yao and G. W. Wornell, "Achieving the full MIMO diversity-multiplexing frontier with rotation-based space-time codes," in *Proc. 2003 Allerton Conf. on Communication, Control and Computing*.

[13] P. Dayal and M. K. Varanasi, "An optimal two transmit antenna space-time code and its stacked extensions," in *Proc. 2003 Asilomar Conf. on Signals, Systems and Computers*, pp. 987–991.

[14] F. Oggier, G. Rekaya, J. C. Belfiore, and E. Viterbo, "Perfect space time block codes," *IEEE Trans. Inf. Theory*, vol. 52, no. 9, pp. 3885–3902, Sept. 2006

[15] P. Elia, B. A. Sethuraman, and P. V. Kumar, "Perfect space-time codes for any number of antennas," *IEEE Trans. Inf. Theory*, vol. 53, no. 11, pp. 3853–3868, Nov. 2007.

[16] B. A. Sethuraman, B. S. Rajan, and V. Shashidhar, "Full-diversity, high-rate space-time block codes from division algebras," *IEEE Trans. Inf. Theory*, vol. 49, no. 10, pp. 2596–2616, Oct. 2003.

[17] K. P. Srinath and B. S. Rajan, "Improved perfect space-time block codes." Available: arXiv:1208.1592v2 [cs.IT], Dec. 2012

[18] Y. Wu and L. Davis, "Fixed-point fast decoding of the Silver code," in *Proc. 2009 IEEE Australian Communication Theory Workshop*, pp. 1–5.

[19] S. K. Mohammed, E. Viterbo, Y. Hong, and A. Chockalingam, "Modulation diversity in fading channels with a quantized receiver," *IEEE Trans. Wireless Commun.*, vol. 11, no. 1, pp. 316–327, Jan. 2012.

[20] Y. Wu, L. Davis, and R. Calderbank, "Finite precision analysis for space-time decoding," *IEEE Trans. Signal Process.*, vol. 57, no. 12, pp. 4861–4870, Dec. 2009.

[21] J. Harshan and E. Viterbo, "Full-rate integer space-time block codes for 2×2 MIMO channels," to appear in *Proc. 2013 IEEE International Conference on Communications*.