

# Polar Codes for Block Fading Channels

Shuiyin Liu, Yi Hong, and Emanuele Viterbo

ECSE Department, Monash University

Melbourne, VIC 3800, Australia

Email: shuiyin.liu, yi.hong, emanuele.viterbo@monash.edu

**Abstract**—In this paper, we consider the design of polar codes for block fading channels. The key idea is to treat the fading process as a *natural polarization*, i.e., the reliability of a symbol in a fading block varies with its fading coefficient. This new viewpoint inspires us to construct polar codes tailored for fading channels by matching code polarization with fading polarization. The resulting codes enjoy an explicit construction, and are shown to provide significant gain with respect to conventional polar BICM schemes and LDPC codes.

## I. INTRODUCTION

The problem of constructing codes for fading channels has received attention in past decades. Conventional approaches assume that the transmitter has no channel knowledge. To spread burst errors in the decoder, the coded bits are interleaved prior to transmission [1–3]. Infinite interleaving is assumed in these techniques to simplify code design and performance analysis. However, infinite interleaving implies infinite encoding/decoding latency and a code optimally designed for infinite interleaving might be actually sub-optimum with finite interleaving. Deriving a coding scheme without bit interleaving is particularly interesting.

One promising research direction is to use the channel state information at the transmitter (CSIT) in the design of codes for fading channels. In general, it is impractical to adapt an arbitrary code to each channel state, but particular structure of polar codes [4], makes this approach feasible. The idea of polar code is to transform a communication channel into polarized subchannels: either completely noisy or noiseless. Information bits are then transmitted over the noiseless subchannels, while fixed or *frozen bits* are sent over the noisy ones. Recently, many efforts have been made to construct polar-based schemes for transmissions over fading channels. In [5], polar lattices for fading channels are constructed. In [6], the author models the subchannels induced by the polarizing transformation as multi-path fading channels and tracks their diversity order and noise variance. In [7], an embedded polar coding scheme is proposed for fading binary symmetric channels. In [8], the author constructs polar codes for block Rayleigh fading channels under the assumption that only channel statistics are available. In [9], the authors study the polarization of block fading channel with two distinct fading coefficients. However, a complex numerical search/optimization procedure is required to design all the above codes. This means that they cannot be

instantaneously available to the transmitter and would only be applied in very slowly varying fading channels.

In this paper, we propose an explicit method to construct polar codes for block fading channels. The low complexity of the code construction allows the implementation of a real-time adaptive coding, valid for both slow and fast fading. Different from the previous approaches in [5–8], our design optimizes the mapping of the coded bits to the fading channels. Since the reliability of each coded bit varies with its fading coefficient, we can treat the fading process as a *natural polarization*. In other words, the proposed polar coding scheme matches code polarization with fading polarization, and thus provides better performance than conventional polar or LDPC BICM [3,10].

Note that the considered problem is related to polar coded modulation [10], where bit-wise mapping are also considered. The authors in [10] focus on optimizing the bit labeling for bit-interleaved coded modulation, while the code design is not updated. In contrast, our scheme combines bit labeling and coding in a single entity. In particular, we adapt the design of polar code (i.e., the design of the frozen set) to the fluctuation of channel reliability.

Section II presents system model. Section III describes the proposed construction of polar codes. Section IV shows simulation results and comparisons with other codes. Section V sets out both theoretical and practical conclusions. The Appendix contains the proof of the theorem.

## II. SYSTEM MODEL AND PROBLEM STATEMENT

### A. System Model

We consider the discrete-time channel model

$$y_i = h_i s_i + n_i, \quad i = 1, \dots, N, \quad (1)$$

where  $N$  is the frame size. In the  $i^{\text{th}}$  channel use,  $s_i = \pm 1$  is the channel input,  $y_i$  is the channel output,  $n_i$  is a zero mean Gaussian noise,  $n_i \sim \mathcal{N}(0, \sigma^2)$ , and  $h_i$  is the channel gain. Here,  $h_i$  and  $n_i$  are assumed to be independent. We don't specify the fading type or the distribution of  $h_i$ , since we assume that the transmitter knows the realization of  $h_i$ .

In the *block fading* channel model, a transmission frame of  $N$  symbols is affected by  $1 \leq B \leq N$  independent fading realizations, resulting in a block of  $m = N/B$  symbols being affected by the same fading realization. Different values  $B$  represents different types of fading, e.g., for  $B = 1$ , we refer to *fast fading* and for  $B = N$  to *slow fading*.

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We assume that each transmitted frame contains one codeword, i.e., in each frame, a rate- $R$  encoder maps  $K$  information bits  $\{u_i\}_1^K$  into  $N$  coded bits  $\{x_i\}_1^N$ , where  $R \triangleq K/N$  is the code rate. Each coded bit is modulated to generate a signal using phase-shift keying (BPSK) modulation, as,

$$s_i = (-1)^{x_i}, \quad i = 1, \dots, N. \quad (2)$$

### B. Polar Codes

1) *Channel Polarization*: Considering that the  $N$  coded bits  $\{x_i\}_1^N$  are transmitted through  $N$  independent copies of binary input memoryless output symmetric channel with transition probabilities  $W(y|x)$ ,  $x \in \mathcal{X}$ ,  $y \in \mathcal{Y}$ , where  $\mathcal{X}$ ,  $\mathcal{Y}$  are input and output alphabets, respectively. We consider the  $N$  transformed binary input channels  $\{W^{(1)}, W^{(2)}, \dots, W^{(N)}\}$  for  $N$  input bits  $\{u_i\}_1^N$ , where

$$W^{(i)} \triangleq W^{(i)}(\mathbf{y}, \{\hat{u}_t\}_1^{i-1} | u_i), \quad (3)$$

represents the transition probability.

Channel polarization is possible if we can find linear transformation from  $\mathbf{u} = [u_1, u_2, \dots, u_N]$  to  $\mathbf{x} = [x_1, x_2, \dots, x_N]$ , denoted by  $\mathbf{x} = \mathbf{u}\mathbf{G}$ , such that each  $u_i$  sees a channel  $W^{(i)}$  of capacity  $I(W^{(i)})$  close to either 1 or 0, as  $N$  goes to infinity. Arikan proved that channel polarization occurs if

$$\mathbf{G} = \mathbf{B}_N \mathbf{F}^{\otimes n}, \quad (4)$$

where  $N = 2^n$ ,  $\mathbf{B}_N$  is a bit-reversal permutation matrix [4],

$$\mathbf{F} = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad (5)$$

and  $\otimes n$  denotes the  $n$  fold Kronecker product of a matrix recursively defined by  $\mathbf{F}^{\otimes n} = \mathbf{F} \otimes \mathbf{F}^{\otimes (n-1)}$ .

The polarized channels  $\{W^{(i)}\}_1^N$  suggest a channel coding scheme, referred to as polar codes, by transmitting a sequence of  $K$  information bits over  $K$  noiseless channels and transmitting a sequence of  $N - K$  fixed or *frozen bits* over  $N - K$  noisy channels.

2) *Polar Encoding*: The encoder of a polar code is

$$\mathbf{x} = \mathbf{u}\mathbf{G} = \mathbf{u}\mathbf{B}_N \mathbf{F}^{\otimes n}. \quad (6)$$

The indices of the input bit sequence  $\{u_i\}_1^N$  are divided into two sets: the information bits set  $\mathcal{A}$  and the frozen bit set  $\mathcal{F}$ . The frozen bits are not decoded given that they are known a priori at receiver. Different from conventional linear block codes, the main challenge for polar code design is finding  $\mathcal{F}$ .

The Bhattacharyya parameter, denoted as  $Z(W)$ , can be used to measure the error probability of a channel  $W$ . For AWGN channels,  $Z(W)$  can be computed as [11]

$$Z(W) = \int_{-\infty}^{+\infty} \sqrt{p(y|0)p(y|1)} dy = \exp\left(-\frac{1}{2\sigma^2}\right). \quad (7)$$

Let  $Z^{(i)}$ ,  $i = 1, \dots, N$ , represent the Bhattacharyya parameter of the bit channel  $W^{(i)}$  in (3). The indices of the  $N - K$  largest values in the set  $\{Z^{(i)}, i = 1, \dots, N\}$  form the frozen bit set  $\mathcal{F}$ . For AWGN channels, the *Gaussian approximation* is used to

compute  $Z^{(i)}$  in [11]. Let  $Z_0^{(i)} = Z(W)$  and  $Z_n^{(i)} = Z^{(i)}$ . The values of  $Z_n^{(i)}$  can be computed recursively from  $Z_0^{(1)}$  as

$$Z_{j+1}^{(i)} = \begin{cases} 2Z_j^{(i)} - (Z_j^{(i)})^2, & 1 \leq i < 2^j + 1 \\ (Z_j^{(i-2^j)})^2, & 2^j + 1 \leq i \leq 2^{j+1} \end{cases} \quad (8)$$

where  $j = 0, 1, \dots, n - 1$ .

3) *Polar Decoding*: Since the frozen bits are already known, the decoder's task reduces to estimating the information bits. To this end, Arikan proposes the *successive cancellation decoding* (SC) procedure summarized in Algorithm 1. In line 5 of Algorithm 1,  $W^{(i)}(\{y_i\}_1^N, \{\hat{u}_t\}_1^{i-1} | u_i)$  represents the likelihood of  $u_i$  given the channel output  $\mathbf{y}$  and the previously decoded bits  $\{\hat{u}_t\}_1^{i-1}$ .

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#### Algorithm 1 Successive Cancellation Decoding [3]

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1: for  $i = 1, 2, \dots, N$  do
2:   if  $i \in \mathcal{F}$  then
3:      $\hat{u}_i \leftarrow u_i$ ; //this is a frozen bit
4:   else
5:      $\hat{u}_i \leftarrow \arg \max_{u_i \in \{0,1\}} W_N^{(i)}(\{y_i\}_1^N, \{\hat{u}_t\}_1^{i-1} | u_i)$ ;
6:   end if
7: end for

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In the SC decoder, past decisions are never revisited in the future. In return for this sub-optimality, the likelihoods  $W^{(i)}(\{y_i\}_1^N, \{\hat{u}_t\}_1^{i-1} | u_i)$  can be computed efficiently and the decoding complexity scales like  $O(N \log N)$ .

To enhance the decoder performance, the list SC decoder was introduced in [12]. The idea is to keep a list of the  $L \geq 1$  most likely codewords at each decoding step. When the last codeword bit has been decoded, the most likely codeword in the list is returned as the decoded codeword. For small block length, the performance of list decoder is very close to the optimal maximum-likelihood (ML) decoder.

### C. Problem Statement

With the received signal being scaled by the channel gain, the block fading channel in (1) can be treated as a set of independent AWGN channels with different received SNRs.

$$\hat{y}_i = (-1)^{x_i} + \hat{n}_i, \quad i = 1, \dots, N, \quad (9)$$

where  $\hat{y}_i = y_i/h_i$ ,  $\hat{n}_i \sim \mathcal{N}(0, \sigma_i^2)$ , and  $\sigma_i^2 = \sigma^2/h_i^2$ .

The error probability of the  $i^{\text{th}}$  channel in (9) is

$$P_E(i) = Q(|h_i|/\sigma), \quad (10)$$

where  $Q(\cdot)$  is the complementary error function. The channels with large fading coefficients are more reliable than the ones with small fading coefficients. In other words, the channels are partially "polarized" by the fading process. This effect, in analogy to code polarization, can be viewed as *fading polarization*.

In this work, we take advantage of fading polarization to construct polar codes. We will optimize the mapping of the coded bits to the fading coefficients, so that the frame error probability is minimized.

### III. CONSTRUCTION OF POLAR CODES FOR BLOCK FADING CHANNELS

We first introduce our design criterion and then derive the optimal mapping between coded bits and fading coefficients.

#### A. Design Criterion

We consider the mapping between the coded bits  $\{x_i\}_1^N$  and the AWGN channels in (9). Without loss of generality, let  $\{x_{\rho(i)}\}_1^N$  be a permuted version of  $\{x_i\}_1^N$  by an index permutation. The received signals in (9) can be written as

$$\hat{y}_i = (-1)^{x_{\rho(i)}} + \hat{n}_i, \quad i = 1, \dots, N. \quad (11)$$

Let us denote the pairs of coded bit and channel SNR as

$$\mathcal{L} = \left\{ (x_{\rho(1)}, h_{\rho(1)}^2/\sigma^2), (x_{\rho(2)}, h_{\rho(2)}^2/\sigma^2), \dots, (x_{\rho(N)}, h_{\rho(N)}^2/\sigma^2) \right\}. \quad (12)$$

Clearly, a coded bit with a different permutation  $\rho$  can observe different SNRs. To simplify notation in the code design, we prefer to use the natural order of coded bits. The channels in (11) and the pairing  $\mathcal{L}$  in (12) can be rewritten as

$$\hat{y}_{\bar{\rho}(i)} = (-1)^{x_i} + \hat{n}_{\bar{\rho}(i)}, \quad i = 1, \dots, N, \quad (13)$$

$$\mathcal{L} = \left\{ (x_1, h_{\bar{\rho}(1)}^2/\sigma^2), (x_2, h_{\bar{\rho}(2)}^2/\sigma^2), \dots, (x_N, h_{\bar{\rho}(N)}^2/\sigma^2) \right\}, \quad (14)$$

where  $\bar{\rho}$  is the inverse permutation of  $\rho$ .

Since the construction of polar codes depends on the pairing  $\mathcal{L}$ , different permutations lead to different performance. Our design criterion is to find the optimal permutation  $\rho$  maximizing the performance of polar codes.

For a given permutation  $\bar{\rho}$ , let  $W^{(i)} : u_i \rightarrow \{\hat{y}_1\}_1^N$  for  $i = 1, \dots, N$ , denote the  $i^{\text{th}}$  bit channel, where

$$W^{(i)} \triangleq W^{(i)}(\{\hat{y}_1\}_1^N, \{u_t\}_1^{i-1} | u_i). \quad (15)$$

Let  $Z_n^{(i)}$  be the Bhattacharyya parameter of  $W^{(i)}$ . The upper bound on the block error probability can be calculated as

$$P_B = \sum_{i \in \mathcal{A}} Z_n^{(i)}, \quad (16)$$

where  $\mathcal{A}$  is the information set corresponding to  $K$  smallest Bhattacharyya parameters. Our code design criterion will be

$$\bar{\rho}_{\text{opt}} = \arg \min_{\bar{\rho}} P_B = \arg \min_{\bar{\rho}} \sum_{i \in \mathcal{A}} Z_n^{(i)}. \quad (17)$$

#### B. Computing $Z_n^{(i)}$

Here we show how to compute  $Z_n^{(i)}$  in (17) for a given  $\bar{\rho}$ . Recalling that the polar encoder contains  $n = \log_2 N$  stages. In each stage, the encoder first pairs the  $N$  bit channels in a predefined order, and then polarizes each pair. Fig. 1 demonstrates the operation of channel polarization: two independent binary channels  $W_1$  and  $W_2$  are selected and transformed to a pair of binary channels  $W'$  and  $W''$ .

Let  $W_j^{(i)}$  denote the  $i^{\text{th}}$  bit channel in stage  $j$ , and  $Z_j^{(i)}$  be the Bhattacharyya parameter of  $W_j^{(i)}$ , where  $j = 0, 1, \dots, n$ .

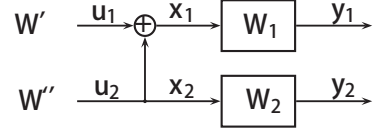


Fig. 1. One level of channel polarization:  $(W_1, W_2) \mapsto (W', W'')$

Note that  $j = 0$  represents the initial stage, i.e., when  $j = 0$ ,  $W_0^{(i)}$  represents the  $i^{\text{th}}$  fading channel

$$\hat{y}_{\bar{\rho}(i)} = (-1)^{x_i} + \hat{n}_{\bar{\rho}(i)}. \quad (18)$$

The values of  $Z_0^{(i)}$  can be calculated by

$$Z_0^{(i)} = \exp\left(-\frac{h_{\bar{\rho}(i)}^2}{2\sigma^2}\right). \quad (19)$$

Note that the values of  $Z_0^{(i)}$  varies with the fading coefficient  $h_{\bar{\rho}(i)}$ , thus is not a constant. It means that in the future encoding stages, we need to polarize bit channels with different reliability. This is different from the standard polar codes design in [4] [11], where channels with the same reliability are polarized. Therefore, the recursion in (8) can not be applied.

In summary, the difficulty of computing  $Z_n^{(i)}$  from  $Z_0^{(i)}$  lies in polarizing bit channels with different Bhattacharyya parameters. This problem is solved by the following lemma.

*Lemma 1:* Suppose that  $(W_1, W_2) \mapsto (W', W'')$  for some set of binary-input channels (Fig. 1). Then,

$$Z(W'') = Z(W_1)Z(W_2), \quad (20)$$

$$Z(W') \leq Z(W_1) + Z(W_2) - Z(W_1)Z(W_2), \quad (21)$$

$$Z(W') \geq Z(W_1) \geq Z(W''). \quad (22)$$

Equality holds in (21) iff  $W_1$  and  $W_2$  are BECs. We have  $Z(W') = Z(W'')$  iff  $Z(W_1) = Z(W_2) = 0$  or 1.

*Proof:* The proof is similar to that of [4, Proposition 5]. ■

Adding (20) and (21) in Lemma 1 shows that the reliability can only improve after channel polarization in the sense that

$$Z(W') + Z(W'') \leq Z(W_1) + Z(W_2), \quad (23)$$

with equality iff  $W_1$  and  $W_2$  are BECs. In this work, we use (20) and (21) to compute  $Z_j^{(i)}$  for  $j > 0$ .

*Example 1:* Let us compute  $Z_1^{(1)}$  and  $Z_1^{(2)}$ .

$$\begin{aligned} Z_1^{(1)} &= Z_0^{(1)} + Z_0^{(2)} - Z_0^{(1)}Z_0^{(2)} \\ &= \exp\left(-\frac{h_{\bar{\rho}(1)}^2}{2\sigma^2}\right) + \exp\left(-\frac{h_{\bar{\rho}(2)}^2}{2\sigma^2}\right) - \exp\left(-\frac{h_{\bar{\rho}(1)}^2 + h_{\bar{\rho}(2)}^2}{2\sigma^2}\right). \end{aligned} \quad (24)$$

$$Z_1^{(2)} = Z_0^{(1)}Z_0^{(2)} = \exp\left(-\frac{h_{\bar{\rho}(1)}^2 + h_{\bar{\rho}(2)}^2}{2\sigma^2}\right). \quad (25)$$

In summary,  $Z_n^{(i)}$  can be obtained from (19), (20) and (21).

### C. Finding $\bar{\rho}_{opt}$

The number of possible permutation patterns in (17) is  $N!/((N/B)!)^B$ , where  $B$  is the number of distinct fading coefficients. For large  $N$ , it is impossible to solve (17) by an exhaustive search. To reduce the search space, we relax the optimization problem in (17) to

$$\bar{\rho}_{opt,1} = \arg \min_{\bar{\rho}} \sum_{i=2,4,\dots,N} Z_1^{(i)}. \quad (26)$$

In other words, we minimize the block error probability of the even channels  $W_1^{(i)}$ ,  $i = 2, 4, \dots, N$ .

*Theorem 1:* Let  $\{h_{\kappa(i)}\}_1^N$  be a sorted version of  $\{h_i\}_1^N$  in ascending order, i.e.,  $h_{\kappa(i)}^2 \leq h_{\kappa(j)}^2$  if  $i \leq j$ . The solution of (26) is

$$\bar{\rho}_{opt,1} = \pi(\Phi), \quad (27)$$

for any arbitrary permutation  $\pi$ , where  $\Phi$  is the set of pairs of indices in the set  $\{\kappa(i)\}_1^N$ , defined by

$$\begin{aligned} \Phi &= \{(\Phi(i, 1), \Phi(i, 2))\}_1^{N/2} \\ &= \{(\kappa(1), \kappa(N)), (\kappa(2), \kappa(N-1)), \dots, (\kappa(N/2), \kappa(N/2+1))\} \end{aligned} \quad (28)$$

i.e., pairing the index of largest fading coefficient with the smallest.

*Proof:* See Appendix A. ■

Theorem 1 shows that the optimal permutation is unique up to a permutation  $\pi$ . In practice, we can always use  $\bar{\rho}_{opt,1} = \Phi$  and the pairing between coded bits and channel SNRs can be written as

$$\mathcal{L}_{opt} = \{(x_1, h_{\kappa(1)}^2/\sigma^2), (x_2, h_{\kappa(N)}^2/\sigma^2), \dots, (x_N, h_{\kappa(N/2+1)}^2/\sigma^2)\} \quad (29)$$

*Example 2:* In Fig. 2, we demonstrate the construction of polar codes with  $N = 8$ . In stage 0, if  $i$  is odd, we map  $x_i$  to the  $\kappa((i+1)/2)$ <sup>th</sup> channel in (13); if  $i$  is even, we map  $x_i$  to the  $\kappa(N+1-i/2)$ <sup>th</sup> channel. Then, we compute

$$Z_0^{(i)} = \begin{cases} \exp\left(-\frac{h_{\kappa((i+1)/2)}^2}{2\sigma^2}\right), & \text{if } i \text{ is odd} \\ \exp\left(-\frac{h_{\kappa(N+1-i/2)}^2}{2\sigma^2}\right), & \text{otherwise} \end{cases} \quad (30)$$

From stage 1 to 3, we compute successively  $Z_1^{(i)}$ ,  $Z_2^{(i)}$ , and  $Z_3^{(i)}$  by using (30), (20) and (21). Finally, we select the frozen bits according to  $Z_3^{(i)}$ .

In summary, the proposed method greatly simplifies the construction of polar codes for fading channels, as no Monte Carlo approach is required. The overall computational complexity is  $O(N \log N)$ . This fact allows us to design polar codes in real time and track the changes in channels.

## IV. SIMULATION RESULTS

This section examines the performance of the proposed polar codes. We consider a block fading channel with frame size 512. In each frame, there are 4 distinct fading coefficients, denoted as,  $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ , i.e., each block of length 128 symbols will be affected by the same fading coefficient. We

fix  $\alpha_1 = 1$ ,  $\alpha_2 = 1/\sqrt{2}$ ,  $\alpha_3 = 1/2$ , and  $\alpha_4 = 1/(2\sqrt{2})$ . We construct (512, 256) polar codes using the mapping proposed in (29). We plot frame error rate (FER) vs. SNR per information bit:

$$\frac{E_b}{N_0} = \frac{E(|h_i s_i|^2)}{R \cdot N_0} = \frac{E(\alpha_j^2)}{\sigma^2} = \frac{0.4688}{\sigma^2}, \quad (31)$$

where  $N_0 = 2\sigma^2$  and  $R = 0.5$ .

For comparison purposes, the performance of (672, 336) LDPC codes from IEEE 802.11ad [13], and standard (512, 256) polar codes with BICM are also shown. For LDPC codes, we extend the frame size to 672. In the polar BICM, the coded bits are random interleaved. Since the interleaved bit channels are approximately equally likely, we chose the design SNR according to (31) and construct the polar codes following [11].

Fig. 3 illustrates the block error rate of proposed polar codes under list SC decoding with different values of list size  $L$ . When  $L = 1$ , list decoding reduces to standard SC decoding. For the LDPC codes, belief propagation decoding with up to 50 iterations was used. With  $L = 4, 16$ , the proposed polar codes provide significant gain with respect to LDPC, even if the LDPC is slightly longer. We find that the standard polar codes with BICM perform quite badly over fading channels. This phenomenon has also been observed in [6]. This loss is due to the mismatch between the code polarization and fading polarization.

In Fig. 4, we compare the performance of polar codes with different mapping between coded bits and fading coefficients. For example, we consider the ‘horizontal’ mapping in [9]

$$\begin{aligned} \mathcal{L}_H &= \{(x_i, h_{\kappa(i)}^2/\sigma^2)\}_1^N \\ &= \{(x_1, h_{\kappa(1)}^2/\sigma^2), (x_2, h_{\kappa(2)}^2/\sigma^2), \dots, (x_N, h_{\kappa(N)}^2/\sigma^2)\}, \end{aligned} \quad (32)$$

where  $h_{\kappa(i)}^2 \leq h_{\kappa(j)}^2$  if  $i \leq j$ . With  $L = 16$ , the polar codes with mapping  $\mathcal{L}_{opt}$  outperform the one with  $\mathcal{L}_H$ . This result confirms that the performance of polar codes is dominated by the mapping between coded bits and fading coefficients. A good mapping can significantly reduce the block error rate.

## V. CONCLUSIONS

In this paper, a method for constructing polar codes for fading channels was proposed. The novelty of our design is to optimize the mapping of the coded bits to the fading channels. In particular, we proposed a new approach to compute the Bhattacharyya parameter for fading channels, which greatly simplifies the construction of polar codes. The simulation results show that with our design, polar codes provide 1.5 dB gain with respect to LDPC codes at block error rate  $10^{-4}$ .

## APPENDIX

### A. Proof of Theorem 1

For simplicity, let

$$\theta(\bar{\rho}(i)) = \exp\left(-\frac{h_{\bar{\rho}(i)}^2}{2\sigma^2}\right). \quad (33)$$

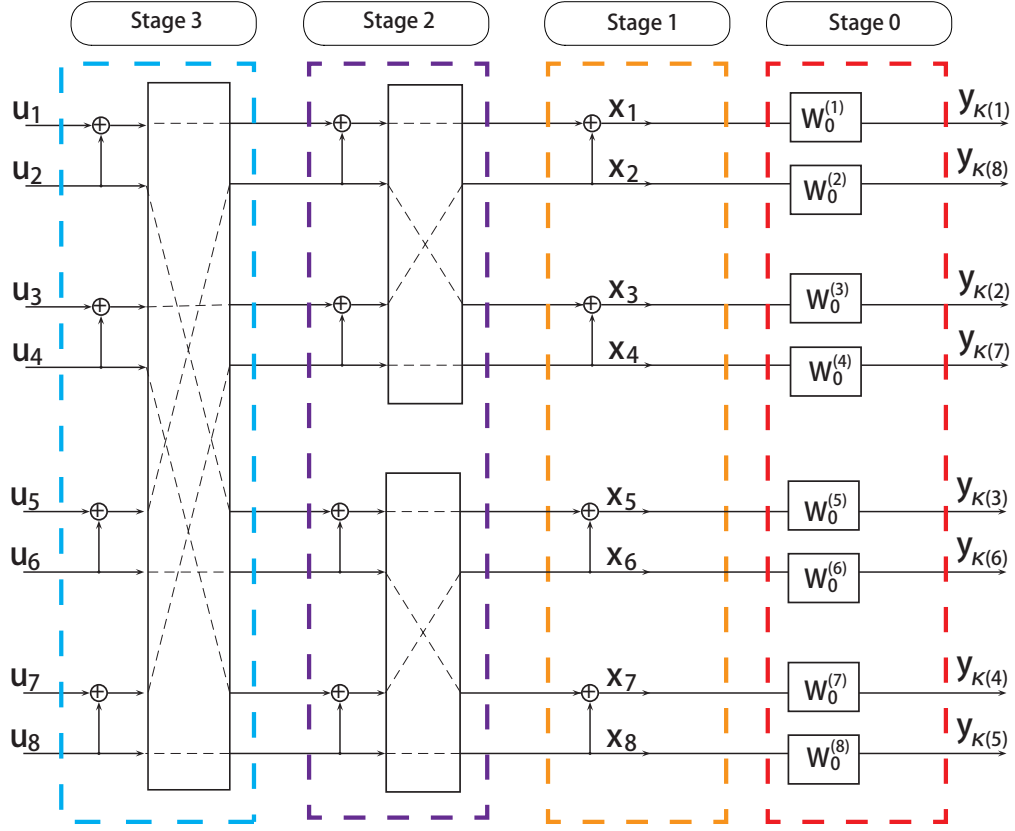


Fig. 2. Construction of polar codes with mapping  $\Phi$  in (28)

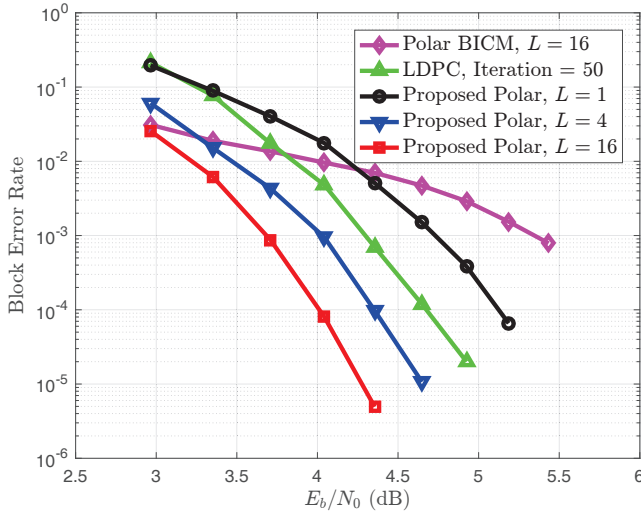


Fig. 3. Performance of different rate-1/2 codes in block fading channels

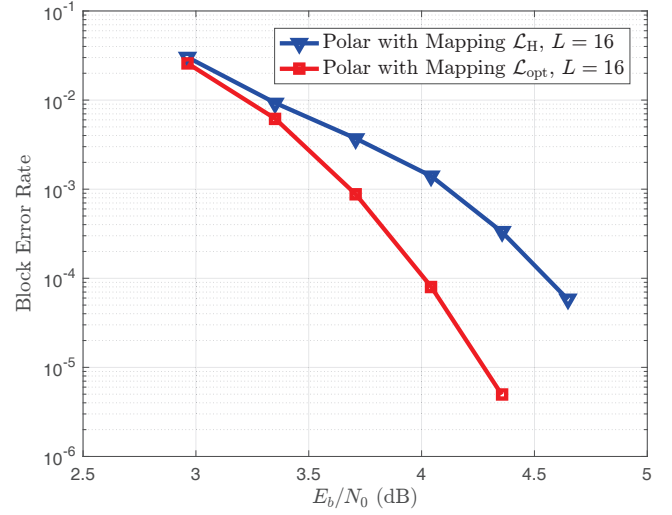


Fig. 4. Comparison of polar codes with different mapping

From (19) and (20), we have

$$\sum_{i=2,4,\dots,N} Z_1^{(i)} = \sum_{i=2,4,\dots,N} \theta(\bar{\rho}(i-1))\theta(\bar{\rho}(i)). \quad (34)$$

We derive the relation

$$\begin{aligned} & \sum_{i=2,4,\dots,N} \theta(\bar{\rho}(i-1))\theta(\bar{\rho}(i)) \\ &= \frac{1}{2} \sum_{i=2,4,\dots,N} [(\theta(\bar{\rho}(i-1))\theta(\bar{\rho}(i)) + \theta(\bar{\rho}(i))\theta(\bar{\rho}(i-1)))] \\ &= \frac{1}{2} \sum_{t=1,2,\dots,N} \theta(t)\theta(\mathcal{P}(t)), \end{aligned} \quad (35)$$

where the sequence  $\{\mathcal{P}(t)\}_1^N$  represents a permutation of  $\{t\}_1^N$ .

Recalling that

$$\theta(\kappa(1)) \geq \theta(\kappa(2)) \geq \dots \geq \theta(\kappa(N)). \quad (36)$$

According to the *rearrangement inequality* in [14], for any permutation  $\mathcal{P}$ , it holds

$$\begin{aligned} & \frac{1}{2} \sum_{t=1,2,\dots,N} \theta(t)\theta(\mathcal{P}(t)) \\ & \geq \frac{1}{2} \sum_{t=1,2,\dots,N} \theta(\kappa(t))\theta(\kappa(N-t+1)) \\ & = \sum_{t=1,2,\dots,N/2} \theta(\kappa(t))\theta(\kappa(N-t+1)). \end{aligned} \quad (37)$$

From (35) and (37), for any permutation  $\bar{\rho}$ , we have

$$\sum_{i=2,4,\dots,N} Z_1^{(i)} \geq \sum_{t=1,2,\dots,N/2} \theta(\kappa(t))\theta(\kappa(N-t+1)). \quad (38)$$

The equality holds when

$$\bar{\rho} = \pi(\Phi), \quad (39)$$

which yields (27). ■

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