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MODULATION SCHEMES DESIGNED FOR THE RAYLEIGH CHANNEL

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ABSTRACT: Usually, the search of better performance in the fading channel leads to the study of error correcting codes or diversity techniques, to be added to the transmission system. These systems are efficiency costly or involve an increase of complexity. In this paper, we take up a modulation approach, and set out to design modulation schemes matched to the Rayleigh fading channel. We present new N-dimensional (N=2,3 and 5) constellations which provide a Nth-order diversity.

1. INTRODUCTION

The error probability of the usual linear modulations (M-PSK or M-QAM) in the fading channel varies as the inverse of the signal to noise ratio [1]. To increase the slope of the error curve, a diversity technique or an error correcting code combined with interleaving can be added. The diversity systems are spectral efficiency costly in case of frequency diversity, or involve additional complexity if multiple antennas are used. From the coding point of view, the Trellis Coded Modulations (TCM) proposed by Ungerboeck [2] in the early eighties are an efficient way of achieving good performance without spectral efficiency loss. Divsalar and Simon [3] give a design criterion of TCM codes in the fading channel, for MPSK modulation. It is currently applied to QPSK and 8-PSK modulation formats. However, the development of digital communications at high data rates, such as the digital HDTV broadcasting, gives rise to an extending demand for more efficient modulation schemes, such as M-QAM modulation. But, the construction of well suited TCM codes becomes a very difficult task when M>16.

In this paper, we consider the design of modulations schemes for the Rayleigh channel. We search at N dimensional lattices which can provide a Nth-order diversity in the fading channel, without the addition of diversity techniques or error correcting codes. The aim is to derive the densest of these lattices, ie the constellations of points with minimum energy for a given error probability. Here we study N=2,3 and 5 dimensions.

Section 2 gives the assumptions. In Sections 3, we derive the two, three and five dimensional lattices. A detection algorithm is given in Section 4. The constellations derived from these lattices are simulated and compared with a M-QAM modulation. The results are presented in Section 5. The conclusion and future prospects are summarized in Section 6.

2. ASSUMPTIONS

2.1 Transmission model

In N dimensions (N≥2), a channel symbol s is denoted by its vector representation $s = (s_1, s_2, \ldots s_N)$, where the coordinates s_i (i=1,N) of s are real. A general baseband transmission model is represented in Figure 1. The mapping of the digital information into a channel symbol is performed by taking a block of k bits and selecting one of $M = 2^k$ signalling symbols. The normalized rate is given by

$$\rho = \frac{2}{N}\log_2 M \quad \text{bits/2-Dim} \qquad (1)$$

An interleaver is added after the mapper to scramble the *components* of the transmitted symbols. Let s denote the transmitted N dimensional real symbol, and z the received signal. Assuming a Rayleigh fading channel and a coherent detection, the coordinates of z are

$$z_i = |\alpha_i| s_i + w_i \qquad i=1,\ldots,N \qquad (2)$$

where w_i is an additive real white Gaussian noise, with zero mean and power density spectrum $N_0/2$, and α_i is a complex Gaussian variable with zero mean and variance γ^2. The envelope of α_i is Rayleigh distributed. Assuming a perfect interleaving, the α_i (i=1,N) are uncorrelated. The signal to noise ratio is equal to

$$\Gamma = \frac{\gamma^2 \overline{E}}{N_0} \qquad (3)$$

where $\overline{E} = 1/(2N)\, E(|s|^2)$ is the average normalized symbol energy. The detection Gaussian metric to minimize is

$$\min_{\hat{s}} \sum_{i=1}^{N} | z_i - |\alpha_i| \hat{s}_i |^2 \qquad (4)$$

2.2 Performance

A Chernoff upper bound on the pairwise error probability is given by

$$P(s \to t) \le \prod_{i=1}^{N} \frac{1}{1 + \frac{\Gamma}{4}\frac{(s_i - t_i)^2}{2\overline{E}}} \qquad (5)$$

At sufficiently high SNR, (5) simplifies to

$$P(s \to t) \le \prod_{i=1}^{N} \frac{1}{\frac{\Gamma}{4}\frac{(s_i - t_i)^2}{2\overline{E}}} \qquad (6)$$

An upper bound of the symbol error probability is then obtained by means of the Union bound

$$P_s \le \frac{1}{M}\sum_{s}\sum_{t \ne s} P(s \to t) \qquad (7)$$

The symbol error probability is dominated by the term in the summation which has the slowest rate of descent with Γ. As an example, consider the two-dimensional (N=2) QAM modulation. There are pairs of distinct symbols with one equal coordinate. The pairwise error probability (5) between these points is proportional to the inverse of the signal to noise ratio. Then, the error probability decreases only inversely with the signal to noise ratio.

Figure now a N-dimensional constellation *where all pairs of distinct symbols have all their coordinates distincts, and the difference is not neglig! ble.* From (5) and (7), it follows that the error rate of the transmission system depicted above decreases inversely with the Nth power of the SNR ie, $P_s \sim 1/\Gamma^N$. In other words, this N-dimensional constellation leads to a Nth-order diversity. As an illustration, consider now the two-dimensions QAM constellation rotated of an angle $\theta \ne 0$ (e.g $\theta = \pi/8$). This is sufficient to hold the above condition and to obtain a second-order diversity. Further examples can be provided in N-dimensions based on this condition. The aim of this paper is to derive the *densest* of these N-dimensional lattices.

2.3 Error distance function

The conditions stated above can be formulated in another way by defining an appropriate distance function between the points, related to the expression of the pairwise error probability in the Rayleigh channel.

Definition 1 : The error distance between two N-dimensional points s and t is defined as

$$\mathrm{dist}_p(s,t) = \sum_{i=1}^{N} \mathrm{Ln}(1+K(s_i-t_i)^2) \qquad (8)$$

with K a parameter. The pairwise error probability (5) between two points is then given by $P(s \rightarrow t) \leq e^{-\mathrm{dist}_p(s,t)}$, with $K = \frac{\Gamma}{4} \frac{1}{2\bar{E}}$.

In the same way, we define an asymptotic error distance related to the asymptotic expression of the pairwise error probability (6).

Definition 2 : The asymptotic error distance between two N-dimensional points s and t is defined as

$$\mathrm{dist}_p^{\infty}(s,t) = \prod_{i=1}^{N} |s_i - t_i| \qquad (9)$$

The optimum constellation can now be defined as the arrangement of points which minimizes the average energy, for a given minimum error distance between the points. In two-dimensions we calculate the densest lattice for the error distance (8). In higher dimensions, the problem becomes very difficult to solve. We consider for simplicity the asymptotic error distance (9), and find some results in the litterature related with theory of numbers, which can be applied to our case.

Remark: We have formulated the problem in a similar way as for the gaussian channel, where the distance to take into account is the euclidean distance.

3. N-DIMENSIONAL ($2 \leq N \leq 5$) LATTICES FOR THE RAYLEIGH CHANNEL

3.1 Two-dimensional lattice for the error distance (8).

The densest two-dimensional lattice is constructed by means of an iterative algorithm. We define first the minimum error distance between the points, denoted r, and calculate the set of points at minimum distance of each other, with minimum average energy. Let $s^1 = (0,0)$ be the starting point, and $s^2, s^3, \ldots, s^m$ the points obtained at step m. The next step consists in finding the point s^{m+1} such that

$$\begin{aligned} &\mathrm{dist}_p(s^1, s^{m+1}) = r \\ &\mathrm{dist}_p(s^j, s^{m+1}) \geq r \qquad \text{for } 2 \leq j \leq m \end{aligned} \qquad (11)$$

and s^{m+1} minimizes the average energy of the set $\{s^1, s^2, s^3, \ldots, s^{m+1}\}$.

The resolution of (11) leads to a regular lattice, i.e the set of points forms an additive discrete group. A generator matrix of this lattice, denoted R_2, is given by

$$R_2 = \begin{pmatrix} a/\sqrt{2} & \frac{a-2b}{2\sqrt{2}} \\ a/\sqrt{2} & \frac{a+2b}{2\sqrt{2}} \end{pmatrix} \qquad (12)$$

where a and b are two scalars which depend on the ratio K defined in section 2.3, and on the minimum given distance r between the points

$$a = \sqrt{\frac{2\left(e^{r/2} - 1\right)}{K}} \qquad (13)$$

$$b = \sqrt{\frac{-5 + e^{r/2} + 4\sqrt{e^r - e^{r/2} + 1}}{2K}}$$

The points of the lattice consist of all points $R_2\eta$, with $\eta = (\eta_1, \eta_2)$ a vector of integer components (see Figure 2). The determinant of this lattice is $d(R_2) = ab$.

3.2 Two-dimensional lattice for the asymptotic error distance.

The result is given in [4, pp.422-432] and is also a regular constellation. A generator matrix of this lattice, denoted R_2^{∞}, is given by

$$R_2^{\infty} = \begin{pmatrix} 1 & \frac{1+\sqrt{5}}{2} \\ 1 & \frac{1-\sqrt{5}}{2} \end{pmatrix} \qquad (14)$$

The determinant is equal to $d(R_2^{\infty}) = \sqrt{5}$.

3.3 Three-dimensional lattice for the asymptotic error distance.

In three dimensions, the lattice is generated by [4, pp.422-432]

$$R_3^{\infty} = \begin{pmatrix} 2\cos\left(\frac{2\pi}{7}\right) & 2\cos\left(\frac{4\pi}{7}\right) & 2\cos\left(\frac{6\pi}{7}\right) \\ 2\cos\left(\frac{4\pi}{7}\right) & 2\cos\left(\frac{6\pi}{7}\right) & 2\cos\left(\frac{2\pi}{7}\right) \\ 2\cos\left(\frac{6\pi}{7}\right) & 2\cos\left(\frac{2\pi}{7}\right) & 2\cos\left(\frac{4\pi}{7}\right) \end{pmatrix} \qquad (15)$$

The determinant is $d(R_3^{\infty}) = 7$.

3.4 Five-dimensional lattice for the asymptotic error distance.

In five dimensions, the lattice is generated by [5, pp.616-625]

$$R_5^{\infty} = \begin{pmatrix} 1 & a & a^2 & a^3 & a^4 \\ 1 & b & b^2 & b^3 & b^4 \\ 1 & c & c^2 & c^3 & c^4 \\ 1 & d & d^2 & d^3 & d^4 \\ 1 & e & e^2 & e^3 & e^4 \end{pmatrix} \qquad (16)$$

where $a=2\cos\left(\frac{\pi}{11}\right)$; $b=2\cos\left(\frac{3\pi}{11}\right)$; $c=2\cos\left(\frac{5\pi}{11}\right)$; $d=2\cos\left(\frac{7\pi}{11}\right)$ and $e=2\cos\left(\frac{9\pi}{11}\right)$.

The determinant is $d(R_5^{\infty})=121$.

4. DETECTION ALGORITHM

In N-dimensions, the detection metric to minimize (4) involves comparing the received signal with M points. For a given normalized rate ρ (1), the value of M increases largely when increasing the dimension N, and involves a very complex detection algorithm at high dimensions. As an example, for ρ=4 bits/2-dim, M=16, M=64 and M=1024, in N=2, N=3 and N=5 dimensions respectively. Hereafter, we describe a simplified detection algorithm, which is available for any constellation. The performance are identical to the performance obtained with the exhaustive detection algorithm.

The decision rule (4) is equivalent to minimize

$$\sum_{i=1}^{N} |\alpha_i|^2 \left| \frac{z_i}{|\alpha_i|} - \hat{s}_i \right|^2 \qquad (17)$$

Stated in another way, this yields to minimize the radius of an ellipsoid centered on $(z_1/|\alpha_1| , \ldots, z_N/|\alpha_N|)$. The detection algorithm consists in incrementing iteratively the radius of the ellipsoid until one point of the constellation is in the ellipsoid. The problem to solve is now, for a given radius, to find the points of the constellation which belong to the interior of the ellipsoid. For simplicity, we determine first the points that belong to the parallelotope which circumscribes the ellipsoid, and then test whether or not they belong to the ellipsoid. For this, we construct a grid which divides the space into small hypercubes. Then, we make a list of points, ordered following which hypercube they fall in. We examine the hypercubes that intersect the parallelotope, and derive the corresponding points from the above list.

The complexity (in number of operations per received point) and the cost (in memory) of this algorithm depend on the size of the hypercubes. If it is too coarse, each hypercube will contain too many points, and if it is too fine, there will be too many hypercubes, most of which will be empty. A compromise is obtained by choosing a grid size so that each hypercube is expected to contain one point in average [6]. That is to say that there are almost as many hypercubes as points. We build then a $G \times G \ldots \times G$ N-dimensional grid with G the nearest integer to $\sqrt[N]{M}$. For example in two-dimensions, we use a 4x4 grid for M=16 points.

A *list* contains the points, ordered according to their place in the grid. In a table of size $G \times G \times \ldots G$, we keep the first point in *list* of each hypercube. If the hypercube contains no point we keep the first point of the next hypercube.

The step incrementing value of the radius of the ellipsoid (17) can be optimized to reduce the number of checked points. We set it to the minimum of $\{1/(2|\alpha_1|), 1/(2|\alpha_2|), \ldots, 1/(2|\alpha_N|)\}$. As an illustration of the complexity of this algorithm, in two-dimensions with a sixteen points constellation, the number of points compared with the received signal during the simulations is in average between 3.1 and 2.0.

5. SIMULATION RESULTS

We simulate the R_2 (with K= 1.6 dB and r=6.7) and the R_N^{∞} (N=2,3 and 5) lattices, with $M=2^{2N}$ points, in the Rayleigh fading channel. The normalized rate of these schemes is ρ=4 bits/2-Dim. Figure 3 gives the symbol error rate P_s versus the bit energy to noise ratio. The results are compared with a 16-QAM modulation. The gains achieved at $P_s=10^{-3}$ are given in table 1. The performance of the 16-R_2 modulation scheme is independent of K and increase when increasing r. However, the optimum is rapidly achieved with a value of r<8. As expected, the N-dimensional modulation schemes present a Nth-order diversity.

	16-R_2	16-R_2^{∞}	64-R_3^{∞}	1024-R_5^{∞}
Gain (dB)	10	10	12	14.7

Table 1. Gains achieved at $P_s=10^{-3}$, compared with a 16-QAM scheme.

6. CONCLUSION AND FUTURE PROSPECTS

In this paper, we present the densest N-dimensional (N=2,3 and 5) regular lattices which provide a N^{th}-order diversity in the Rayleigh fading channel. For a normalized rate of ρ = 4 bits/2-Dim, the gain is in the range of 10 to 14.7 dB, at a symbol error rate of 10^{-3}, compared to the 16-QAM. Also, we describe a simplified detection algorithm, which provides the same performance as the exhaustive one. A similar analysis can be conducted in high-dimensional spaces in order to obtain higher diversity orders. Also, the addition of a well suited TCM code can improve the performance.

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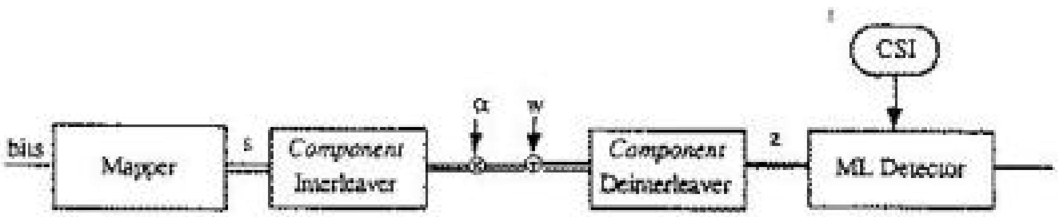

Figure 1. Baseband transmission model

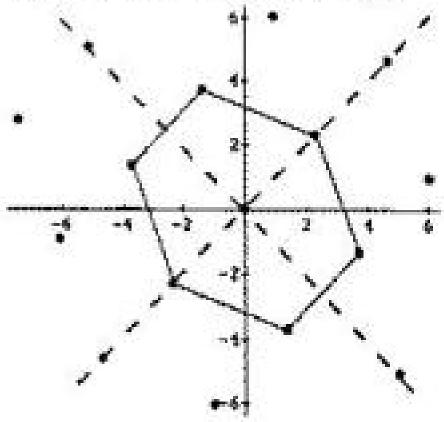

Figure 2. R2 constellation

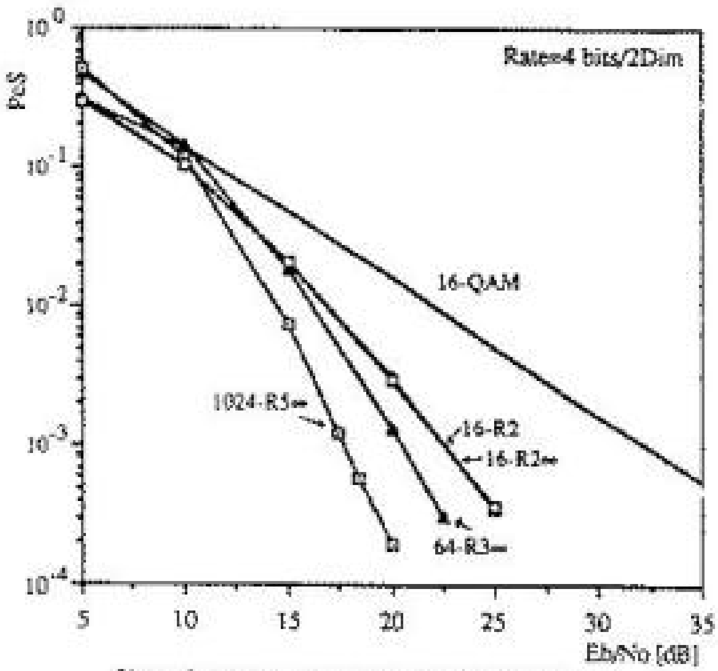

Figure 3. N dimensional (N=2,3,5) constellations
in the Rayleigh channel