# Performance of High Diversity Multidimensional Constellations for the Fading Channel

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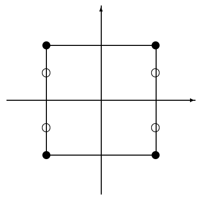
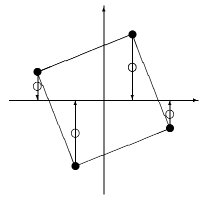


## Introduction



- High diversity multidimensional constellations offer a new type of diversity technique called *modulation diversity* which enhance the error performance of communication systems over the Rayleigh fading channel.
- The modulation diversity order of a signal set is defined as the minimum number of *different* components between any two distinct points of the set.
- Modulation diversity can also be viewed as a special form of signal space coding producing highly efficient (in terms of bandwidth and power) schemes over the fading channel.



## Channel model

Real vector independent Rayleigh fading channel

$$\mathbf{y} = \mathbf{g} \odot \mathbf{x} + \mathbf{n} \qquad y_i = g_i x_i + n_i \quad i = 1, \dots n$$

- $\mathbf{x} = (x_1, x_2, \dots x_n)$ transmitted symbol from a given $n$-dimensional constellation $S$.
- $\mathbf{n} = (n_1, n_2, \dots n_n)$: additive white Gaussian noise vector $n_i = \mathcal{N}(0, N_0/2)$.
- $\mathbf{g} = (g_1, g_2, \dots g_n)$: independent real Rayleigh distributed fading coefficients with $E[g_i] = 1$.
- $\mathbf{y} = (y_1, y_2, \dots y_n)$: received symbol.
- Perfect phase tracking and channel state information (CSI) are assumed at the receiver, which performs maximum-likelihood (ML) detection by minimizing
$$m(\mathbf{x}|\mathbf{y}, \mathbf{g}) = ||\mathbf{y} - \mathbf{g} \odot \mathbf{x}||^2 \ \ \forall \ \ \mathbf{x} \in S.$$



## Chernoff Bound

- Union Bound
$$P(e) \le \frac{1}{|S|} \sum_{\mathbf{x} \in S} \sum_{\mathbf{x} \ne \hat{\mathbf{x}}} P(\mathbf{x} \to \hat{\mathbf{x}}) \ .$$
- Pairwise error probability (PEP) $P(\mathbf{x} \to \hat{\mathbf{x}})$ is commonly approximated by using the Chernoff bound
$$P(\mathbf{x} \to \hat{\mathbf{x}}) \le \prod_{i=1}^{L} \frac{1}{(1 + |x_i - \hat{x}_i|^2/4N_0)} = \frac{4^L}{[d_p^{(L)}]^2} N_0^L + O(N_0^{L+1}) \ .$$
- $L$ is the modulation diversity order of the signal set is defined as the minimum number of *different* components between any two distinct points of the set.
- $d_p^{(L)} = \Pi_{x_i \ne \hat{x}_i} |x_i - \hat{x}_i|$ is the *$L$-product distance* between $\mathbf{x}$ and $\hat{\mathbf{x}}$.
- Good signal sets have high $L$ and $d_{p,\min}^{(L)}$ and were first contructed using Algebraic Number Theory by Boullé and Belfiore in 1992.

## Exact PEP

**Theorem.** *Let* $\mathbf{x} = (x_1, x_2, \dots x_n)$, $\widehat{\mathbf{x}} = (\widehat{x}_1, \widehat{x}_2, \dots \widehat{x}_n)$, $\delta_i = |x_i - \widehat{x}_i| \neq 0$ *for* $i = 1, 2, \dots n$ *and assume that all* $\delta_i$*'s are distinct, then the pairwise error probability is then given by*

$$P(\mathbf{x} \to \widehat{\mathbf{x}}) = \frac{1}{2} \sum_{i=1}^{n} \left(1 - \frac{\delta_i}{\sqrt{4N_0 + \delta_i^2}}\right) \prod_{\substack{j=1 \\ j \neq i}}^{n} \frac{\delta_i^2}{\delta_i^2 - \delta_j^2} \ .$$

**Sketch of the proof.**

- $$P(\mathbf{x} \to \widehat{\mathbf{x}}) = P\left(\sum_{i=1}^{n} |y_i - g_i \widehat{x}_i|^2 < \sum_{i=1}^{n} |y_i - g_i x_i|^2\right) = P(\Delta < 0)$$
  where $\quad \Delta = \sum_{i=1}^{n} |g_i|^2 |x_i - \widehat{x}_i|^2 - 2\, g_i (x_i - \widehat{x}_i) n_i \ .$
- Laplace inversion formula: $\quad P(\mathbf{x} \to \widehat{\mathbf{x}}) = \frac{1}{2\pi j} \int_{c - j\infty}^{c + j\infty} \Phi_\Delta(s) \frac{ds}{s} \ .$

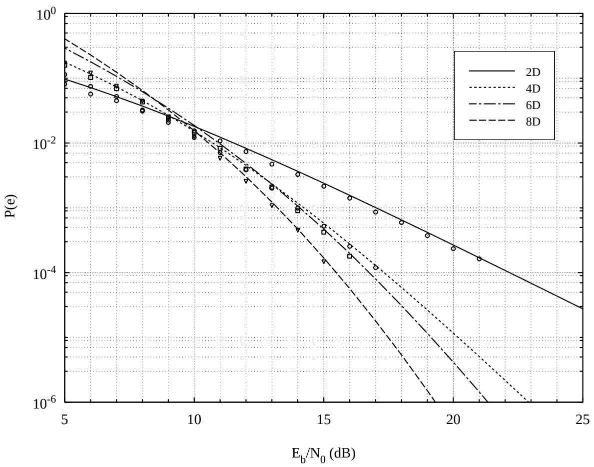


## Remarks

- If $L < n$, with the previous notations, let $\delta_i$ for $i = 1, \dots, L$ be the non zero elements, then we have
  $$P(\mathbf{x} \to \widehat{\mathbf{x}}) = \frac{1}{2} \sum_{i=1}^{L} \left(1 - \frac{\delta_i}{\sqrt{4N_0 + \delta_i^2}}\right) \prod_{\substack{j=1 \\ j \neq i}}^{L} \frac{\delta_i^2}{\delta_i^2 - \delta_j^2}$$
- The singular points when $\delta_i = \delta_j$ are only apparent and can be eliminated by a small perturbation on the $\delta$ without affecting significantly the final result.
- As an example we consider the rotated cubic constellations proposed by Boutros and Viterbo 1996.

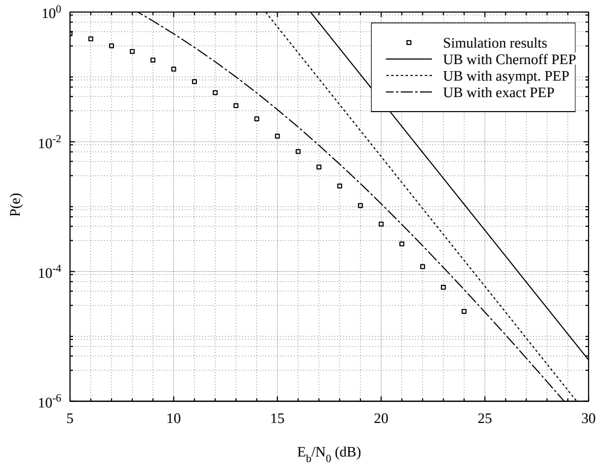


## Rotated hypercube constellations with $n = 2, 4, 6, 8$






## Rotated four-dimensional 4-PAM



## High Diversity ⇒ AWGN

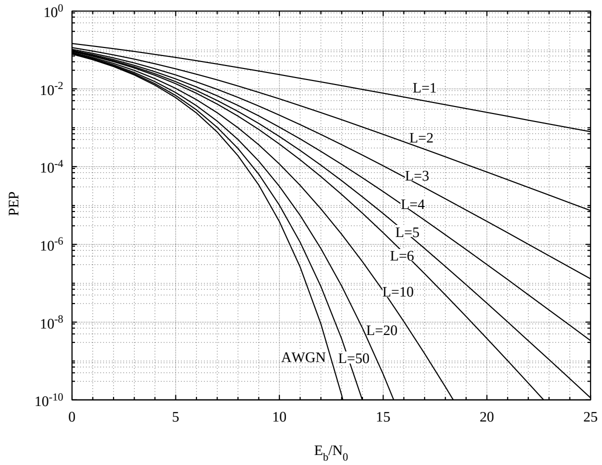
- The minimum PEP under the constraint of constant $\Sigma_{i=1}^{L}\delta_i^2$ is obtained when all $\delta_i = \delta$ are equal.

$$P(\mathbf{x} \to \hat{\mathbf{x}}) = \frac{1}{2} - \frac{1}{2}(1 + 4N_0/\delta^2)^{-L+1/2} \sum_{i=0}^{L-1} \binom{L-1/2}{i} (4N_0/\delta^2)^i$$

The two-point constellation $\pm(\delta/2, \ldots, \delta/2)$ is optimal for $L = n$.



## High Diversity ⇒ AWGN






## Conclusions

- We presented an exact expression of the PEP for component-interleaved multidimensional schemes for the Rayleigh fading channel.
- This provides better approximation of the error performance than obtained by using the Chernoff bound.
- The proposed bound can be used for more accurate analysis and design of high-diversity multidimensional constellations for the fading channel.