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Radius of Convergence for Volterra Series Transfer Function in Modeling \Single-Mode Fiber

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OPTICAL POWER LIMIT FOR LINEAR OPERATION IN OPTICAL FIBER TRANSMISSION SYSTEM VIA VOLTERRA SERIES TRANSFER FUNCTION APPROACH

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Abstract

An optical transfer function of optical fibers in an explicit frequency domain facilitates the modelling and design of optical communications systems. Due to the nonlinearity of the optical fibres operating under high average total optical power, the Volterra series technique is essential.

A convergence criterion for Volterra Series Transfer Function (VSTF) approach for solving the nonlinear Schrodinger (NLS) wave equation representing the propagation of an optical pulse over a dispersive and nonlinear optical fibre is presented. The VSTF method is used to solve the NLS wave equation for single-mode fiber by approximating the solution with a generalized Taylor's series in its frequency domain representation form by taking the multi-dimensional Fourier transform. It is shown that the radius of convergence of VSTF is closely corresponding to the maximum power limit that can be supported by silica fibre so that it is still operating in the linear regime.

Index terms – Volterra series, optic fiber transmission systems

I. Introduction

Fiber-optic communication is the fastest growing areas in modern communication systems in order to satisfy the intensive demand on bandwidth by the Internet. Systems of higher bandwidth and longer transmission distance and multiplexed optical carriers have been installed at a tremendous pace. With the advent of erbium-doped fiber amplifiers (EDFAs) since 1989, the major limitation of fiber-optic communication systems has changed from loss-limited to dispersion-limited. The increase in bandwidth and bit rates by high speed modulation and multiplexing of optical carriers in the S-, C- and L-bands and transmission distance further complicates the design of optical communications systems. It is thus essential to develop a more efficient model to represent the communications channel using optical fibers.

Further due to the increase of the total number of optical channels, ie. the multiplexed optical information-modulated carriers, the fibre channel reaches its nonlinear regime and hence a frequency domain approach in modelling fibres is normally avoided by employing the well-known split-step Fourier (SSF) procedures. Therefore the principal difficulty in modelling a single-mode optical fiber as a transmission channel lies in an effective representation of the self-induced nonlinear effects. Nonlinearities existing in fibers, optical amplifiers and other devices have long been recognized as one of the major limitations for optical communication systems. Various nonlinear effects such as self-phase modulation (SPM), cross-phase modulation (CPM), stimulated Raman scattering (SRS), stimulated Brillouin scattering (SRS) and four-wave mixing (FWM) have been known to cause serious problems in long-haul-high-speed optical networks. The widely used Split-step Fourier (SSF) method incorporates the fiber nonlinear effects in the time domain into the calculation of linear effects in the frequency domain via an intermediate step where the optical fields have been Fourier transformed into the frequency domain[7]. Despite the high efficiency of SSF method, it is a recursive method and hence can be unwieldy in designing optical communication systems.

The Volterra series transfer function (VSTF) is a mathematically tool that is used to approximate the nonlinear effects in a system [1]. It has been used to overcome the nonlinearity effects in several engineering nonlinear problems for its intuitive representation of the nonlinear elements. Unlike other numerical methods, the VSTF method is non-recursive. An application of the Volterra series for optical communication system first appeared in literature in 1997 [2], and since then a number of papers have been published on this topic [3] [4]. These works have, however, focused on how to apply the VSTF method to solve the nonlinear Schrondinger (NLS) equation for the optical communication systems, and have compared the accuracy of the VSTF method with the more traditionally used methods such as the SSF approach. However, none of the papers have discussed the issue of the validity of the VSTF method in representing the nonlinearity of the optical fibre channel, that is the limits of the convergence of VSTF. Besides, it is also important to estimate accurately the upper limit of the input optical power to be applicable in VSTF model. Indeed the radius of convergence of the VSTF model dictates the convergence of the series and correlates to this limit. In this paper, we present the effects of the convergence of VSTF and its applications in the modelling of optical communication systems and most importantly the upper limit for convergence and its physical relationship with the possible maximum power that can be transmitted through via fibre for linear operation. The paper is organised as follows. A brief representation of the NLS equation is given in the next section followed by its representation by the Volterra series in section 3. Numerical results and its corrresponding physical interpretation are given in Section 4. The impacts of such findings of the convergence of the Volterra series on optical communications systems are also described.

II. FIBER NONLINEARITIES – NONLINEAR SCHRODINGER (NLS) EQUATION

As in most nonlinear systems, nonlinear effects in optical communication systems increase with an increase in the input power level, as well as with increase in the fiber length. The generalized nonlinear Schrodinger equation is used to model all the known linear and nonlinear effects in optical fibers

$$\frac{\partial A}{\partial z} + \frac{\alpha_0}{2} A + \beta_1 \frac{\partial A}{\partial t} + j\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2} + \frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3} = j\gamma |A|^2 A - a_1 \frac{\partial\left(|A|^2 A\right)}{\partial t} + a_2 \frac{\partial A}{\partial t}|A|^2 + a_3 \frac{\partial |A|^2}{\partial t} A$$

$$+ jQ_R A \int_{-\infty}^{\infty} s_r(t - t_1) |A(t_1, z)|^2 dt_1 \qquad (1)$$

where $A = A(t,z)$ represents the slowly-varying complex envelope of the input pulse, α is the linear attenuation coefficient of the optical fiber; $\beta_1 = \frac{1}{v_g}$ is the inverse of the optical carrier group velocity v_g; β_2 is the second-order dispersion parameter and is usually termed as the group velocity dispersion (GVD); β_3 is the third-order dispersion parameter; $\gamma = \frac{\omega_0}{cA_{eff}} n_2$, $a_1 = \frac{a_0}{\omega_0}$, a_2 and a_3 are the nonlinear coefficients of the single-mode fiber, where c is the speed of light, A_{eff} is the effective cross-section of the fiber, and n_2 is the refractive index of the core section; $Q_R = \frac{\omega_0}{cA_{eff}} G_R$, is the Raman constant, with G_R as the Raman gain coefficient factor; $s_R(t)$ is the time-domain representation of the Raman gain spectrum, $S_R(\omega)$. We note here that the frequency component ω_o is assigned as the central optical frequency of the carrier under consideration, while all other frequency components ω_n (as described later) are generalised including all other adjacent carriers or components affecting the transmission channel. The NLS equation is in fact evaluated all the signal frequency components over the signal base-band, ie. the optical spectrum including an optical carrier and its base-band information spectrum has been shifted to the origin.

The NLS equation cannot be solved analytically due to its nonlinear nature. Normally, a complete numerical method such as split-step Fourier (SSF) method or Runga-Kutta method are used to reach the final the solution, the output pulses at the end of the transmission fibre. Under these numerical methods, the fibre is divided into small segments and the waveform at each segment is computed sequentially. The length of each segment has to be kept small to ensure accuracy of these numerical methods, and this could become very computationally extensive when the length of the transmission fibre increases. Further more numerical methods are difficult in designing a complete system by interconnection of different transfer fnction since it does not provide sufficient information of the system's characteristic. An important aspect and advantage of representing the optical transmission channel in the frequency domain is that any optical component can be inserted into the optical transmission system can be modelled and analysed by inspection and hence reducing the analysis cycle time.

III. Fibre Transmission Model by Volterra Series Transfer Function

The weakness of most of the recursive methods in solving the NLS is that they do not provide much useful information to help the engineers to characterise the nonlinear effects. The

Volterra series model provides an elegant way of describing a system's nonlinearities, and enables the designers to see clearly where and how the non-linearity affects the system performance. Although Refs.[2-4] have given an outline of the kernels of the transfer function using the Volterra series, we believe that it is necessary for clarity and physical representation of these functions, brief derivations of the nonlinear transfer functions of an optical fibre operating under nonlinear conditions, ie. significant optical pulse energy.

The Volterra series transfer function of a particular optical channel can be obtained in the frequency-domain as a relationship between the input spectrum $X(\omega)$ and the output spectrum $Y(\omega)$, as

$$Y(\omega)=\sum_{n=1}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}H_n(\omega_1,\cdots,\omega_{n-1},\omega-\omega_1-\cdots-\omega_{n-1})\times X(\omega_1)\cdots X(\omega_{n-1})X(\omega-\omega_1-\cdots-\omega_{n-1})d\omega_1\cdots d\omega_{n-1} \tag{2}$$

where $H_n(\omega_1,\cdots,\omega_n)$ is the *n*th-order frequency domain Volterra kernel including all signal frequencies of orders 1 to n. The wave propagation inside a single-mode fibre can be governed by a simplified version of the NLS wave equation [1] with only the self phase modulation effect is included as

$$\frac{\partial A}{\partial z}=-\frac{\alpha_0}{2}A-\beta_1\frac{\partial A}{\partial t}-j\frac{\beta_2}{2}\frac{\partial^2 A}{\partial t^2}-\frac{\beta_3}{6}\frac{\partial^3 A}{\partial t^3}+j\gamma|A|^2 A \tag{3}$$

where $A=A(t,z)$. The proposed solution of the NLS equation can be written with respect to the VSTF model of up to 5$^{\text{th}}$ order as

$$A(\omega,z)=H_1(\omega,z)A(\omega)+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}H_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2,z)\times A(\omega_1)A^*(\omega_2)A(\omega-\omega_1+\omega_2)d\omega_1 d\omega_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega-\omega_1+\omega_2-\omega_3+\omega_4,z)\ \times A(\omega_1)A^*(\omega_2)A(\omega_3)A^*(\omega_4)$$

$$\times A(\omega-\omega_1+\omega_2-\omega_3+\omega_4)d\omega_1 d\omega_2 d\omega_3 d\omega_4 \tag{4}$$

where $A(\omega)=A(\omega,0)$, that is the amplitude envelop of the optical pulses at the input of the fibre. Taking the Fourier transform of (3) and assuming $A(t,z)$ is of sinusoidal form we have

$$\frac{\partial A(\omega,z)}{\partial z}=G_1(\omega)A(\omega,z)\ +\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2)A(\omega_1,z)A^*(\omega_2,z)\ \times A(\omega-\omega_1+\omega_2,z)d\omega_1 d\omega_2 \tag{5}$$

where $G_1(\omega)=-\frac{\alpha_0}{2}+j\beta_1\omega+j\frac{\beta_2}{2}\omega^2-j\frac{\beta_3}{6}\omega_3$ and $G_3(\omega_1,\omega_2,\omega_3)=j\gamma$. ω is taking the values over the signal bandwidth ad beyond in overlapping the signal spectrum of other optically modulated carriers while $\omega_1..\omega_3$ are all also taking values over similar range as that of ω. For general expression the limit of integration is indicted ver the entire range to infinitive.

Substituting (2) into (3) and equate both sides, the kernels can be obtained after some algebraic manipulations

$$\frac{\partial}{\partial z}\Big[H_1(\omega,z)A(\omega)+\int_{-\infty}^{\infty}\int_{-\infty i}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2,z)\times A(\omega_1)A^*(\omega_2)A(\omega-\omega_1+\omega_2)d\omega_1 d\omega_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega-\omega_1+\omega_2-\omega_3+\omega_4,z)\ \times A(\omega_1)A^*(\omega_2)A(\omega_3)A^*(\omega_4)$$

$$\times A(\omega-\omega_1+\omega_2-\omega_3+\omega_4)d\omega_1 d\omega_2 d\omega_3 d\omega_4\Big]$$

$$=G_1(\omega)\Big[H_1(\omega,z)A(\omega)+\int_{-\infty}^{\infty}\int_{-\infty i}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2,z)\times A(\omega_1)A^*(\omega_2)A(\omega-\omega_1+\omega_2)d\omega_1 d\omega_2$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega-\omega_1+\omega_2-\omega_3+\omega_4,z)\times A(\omega_1)A^*(\omega_2)A(\omega_3)A^*(\omega_4)$$

$$\times A(\omega-\omega_1+\omega_2-\omega_3+\omega_4)d\omega_1 d\omega_2 d\omega_3 d\omega_4\Big] + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2)$$

$$\times\Big[H_1(\omega_1,z)A(\omega_1)+\int_{-\infty}^{\infty}\int_{-\infty i}^{\infty} H_3(\omega_{11},\omega_{12},\omega_1-\omega_{11}+\omega_{12},z)\times A(\omega_{11})A^*(\omega_{12})A(\omega_1-\omega_{11}+\omega_{12})d\omega_{11}d\omega_{12}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_{11},\omega_{12},\omega_{13},\omega_{14},\omega_1-\omega_{11}+\omega_{12}-\omega_{13}+\omega_{14},z) \quad \times A(\omega_{11})A^*(\omega_{12})A(\omega_{13})A^*(\omega_{14})$$

$$\times A(\omega_1-\omega_{11}+\omega_{12}-\omega_{13}+\omega_{14})d\omega_{11}d\omega_{12}d\omega_{13}d\omega_{14}\Big]\times\Big[H_1(\omega_2,z)A(\omega_2)+\int_{-\infty}^{\infty}\int_{-\infty i}^{\infty} H_3(\omega_{21},\omega_{22},\omega_2-\omega_{21}+\omega_{22},z)$$

$$\times A(\omega_{21})A^*(\omega_{22})A(\omega_2-\omega_{21}+\omega_{22})d\omega_{21}d\omega_{22}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_{21},\omega_{22},\omega_{23},\omega_{24},\omega_2-\omega_{21}+\omega_{22}-\omega_{23}+\omega_{24},z) \quad \times A(\omega_{21})A^*(\omega_{22})A(\omega_{23})A^*(\omega_{24})$$

$$\times A(\omega_2-\omega_{21}+\omega_{22}-\omega_{23}+\omega_{24})d\omega_{21}d\omega_{22}d\omega_{23}d\omega_{24}\Big]^*\times\big[H_1(\omega-\omega_1+\omega_2,z)A(\omega-\omega_1+\omega_2)$$

$$+\int_{-\infty}^{\infty}\int_{-\infty i}^{\infty} H_3(\omega_{31},\omega_{32},\omega-\omega_1+\omega_2-\omega_{31}+\omega_{32},z) \times A(\omega_{31})A^*(\omega_{32})A(\omega-\omega_1+\omega_2-\omega_{31}+\omega_{32})d\omega_{31}d\omega_{32}$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_{31},\omega_{32},\omega_{33},\omega_{34}\ ,\omega-\omega_1+\omega_2-\omega_{31}+\omega_{32}-\omega_{33}+\omega_{34},z)\times A(\omega_{31})A^*(\omega_{32})A(\omega_{33})A^*(\omega_{34})$$

$$\times A(\omega-\omega_1+\omega_2-\omega_{31}+\omega_{32}-\omega_{33}+\omega_{34}) \times d\omega_{31}d\omega_{32}d\omega_{33}d\omega_{34}\big] \tag{6}$$

Equating the 1st order terms on both sides we obtain

$$\frac{\partial}{\partial z}H_1(\omega,z)=G_1(\omega)H_1(\omega,z) \tag{7}$$

Thus the solution for the first order transfer function (7) is then given by

$$H_1(\omega,z)=e^{G_1(\omega)z}=e^{\left(-\frac{\alpha_0}{2}+j\beta_1\omega+j\frac{\beta_2}{2}\omega^2-j\frac{\beta_3}{6}\omega^3\right)z} \tag{8}$$

This is in fact the linear transfer function of an optical fibre with the dispersion factors β_2 and β_3. Similarly for the 3[rd] order terms we have

$$\frac{\partial}{\partial z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2,z)\times A(\omega_1)A^*(\omega_2)A(\omega-\omega_1+\omega_2)d\omega_1 d\omega_2$$

$$=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} G_3(\omega_1,\omega_2,\omega-\omega_1+\omega_2)H_1(\omega_1,z)A(\omega_1)H_2^*(\omega_2,z)\times A(\omega_2)H_1(\omega-\omega_1+\omega_2)A(\omega-\omega_1+\omega_2)d\omega_1 d\omega_2 \tag{9}$$

Now letting $\omega_3=\omega-\omega_1+\omega_2$ then it follows

$$\frac{\partial H_3(\omega_1,\omega_2,\omega_3,z)}{\partial z}=G_1(\omega_1-\omega_2+\omega_3)H_3(\omega_1,\omega_2,\omega_3,z)+G_3(\omega_1,\omega_2,\omega_3)H_1(\omega_1,z)H_1^*(\omega_2,z)H_1(\omega_3,z) \tag{10}$$

The 3[rd] kernel transfer function can be obtained as

$$H_3(\omega_1,\omega_2,\omega_3,z)=G_3(\omega_1,\omega_2,\omega_3)\times\frac{e^{(G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3))z}-e^{G_1(\omega_1-\omega_2+\omega_3)z}}{G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3)-G_1(\omega_1-\omega_2+\omega_3)} \tag{11}$$

The 5th order kernel can similarly be obtained as

$$\begin{aligned}
&H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega_5,z)\\
&=\frac{H_1(\omega_1,z)H_1^*(\omega_2,z)H_1(\omega_3,z)H_1^*(\omega_4,z)H_1(\omega_5,z)-H_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5,z)}{G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3)+G_1^*(\omega_4)+G_1(\omega_5)-G_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5)}\\
&\times\left[\frac{G_3(\omega_1,\omega_2,\omega_3-\omega_4+\omega_5)G_3(\omega_3,\omega_4,\omega_5)}{G_1(\omega_3)+G_1^*(\omega_4)+G_1(\omega_5)-G_1(\omega_3-\omega_4+\omega_5)}+\frac{G_3(\omega_1,\omega_2-\omega_3+\omega_4,\omega_5)G_3^*(\omega_2,\omega_3,\omega_4)}{G_1^*(\omega_2)+G_1(\omega_3)+G_1^*(\omega_4)-G_1^*(\omega_2-\omega_3+\omega_4)}\right.\\
&\left.+\frac{G_3(\omega_1-\omega_2+\omega_3,\omega_4,\omega_5)G_3(\omega_1,\omega_2,\omega_3)}{G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3)-G_1(\omega_1-\omega_2+\omega_3)}\right]-\frac{G_3(\omega_1,\omega_2,\omega_3-\omega_4+\omega_5)G_3(\omega_3,\omega_4,\omega_5)}{G_1(\omega_3)+G_1^*(\omega_4)+G_1(\omega_5)-G_1(\omega_3-\omega_4+\omega_5)}\\
&\times\frac{H_1(\omega_1,z)H_1^*(\omega_2,z)H_1(\omega_1-\omega_2+\omega_3,z)-H_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5,z)}{G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3-\omega_4+\omega_5)-G_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5)}\\
&-\frac{G_3(\omega_1,\omega_2-\omega_3+\omega_4,\omega_5)G_3^*(\omega_2,\omega_3,\omega_4)}{G_1^*(\omega_2)+G_1(\omega_3)+G_1^*(\omega_4)-G_1^*(\omega_2-\omega_3+\omega_4)}\\
&\times\frac{H_1(\omega_1,z)H_1^*(\omega_2-\omega_3+\omega_4,z)H_1(\omega_5,z)-H_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5,z)}{G_1(\omega_1)+G_1^*(\omega_2-\omega_3+\omega_4)+G_1(\omega_5)-G_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5)}
\end{aligned}$$

$$-\frac{G_3(\omega_1-\omega_2+\omega_3,\omega_4,\omega_5)G_3(\omega_1,\omega_2,\omega_3)}{G_1(\omega_1)+G_1^*(\omega_2)+G_1(\omega_3)-G_1(\omega_1-\omega_2+\omega_3)}$$
$$\times\frac{H_1(\omega_1-\omega_2+\omega_3,z)H_1^*(\omega_4,z)H_1(\omega_5,z)-H_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5,z)}{G_1(\omega_1-\omega_2+\omega_3)+G_1^*(\omega_4)+G_1(\omega_5)-G_1(\omega_1-\omega_2+\omega_3-\omega_4+\omega_5)} \qquad (12)$$

Higher order terms can be derived with ease if higher accuracy is required. However in practice such higher order would not exceed the 5th rank. We can understand that for a length of a uniform optical fibre the 1st to nth order frequency spectrum transfer can be evaluated indicating the linear to nonlinear effects of the optical signals transmitting through it. Indeed the 3rd and 5th order kernel transfer functions based on the Volterra series indicate the optical filed amplitude of the frequency components which contribute to the distortion of the propagated pulses. An inverse of these higher order functions would give the signal distortion in the time domain. Thus the VSTFs allow us to conduct distortion analysis of optical pulses and hence an evaluation of the bit-error-rate of optical fibre communications systems. We will present these studies in future works.

The superiority of such Volterra transfer function expressions allow us to evaluate each effects individually, especially the nonlinear effects so that we can design and manage the optical communications systems under linear or nonlinear operations. Currently this linear-nonlinear boundary of operations is critical for system implementation, especially for optical systems operating at 40 Gbps where linear operation and carrier suppressed return-to-zero format is employed. As a norm in series expansion the series need to be converged to a final solution. It is this convergence that would allow us to evaluate the limit of non-linearity in a system. The issues involved the convergence and its relation ship with linearity and non-linearity are therefore presented in the next section.

IV. CONVERGENCE PROPERTY OF VSTF

As we can see from previous section that the Volterra series transfer function takes the form of a power series, whose convergence can be examined with a number of well-established tests. The ratio test is chosen in this paper to test the convergence of the VSTF as it would lead to the best estimation of the convergence of the series. An infinite Volterra series can be represented by a function of all frequency components as

$$\sum_{n=1}^{\infty} Y_n(\omega_1,\cdots,\omega_n) = \sum_{n=1}^{\infty} H_n(\omega_1,\cdots,\omega_n) U(\omega_1)\cdots U(\omega_n) \le \sum_{n=1}^{\infty} |H_n(\omega_1,\cdots,\omega_n)| |U_{\max}|^n \tag{13}$$

where $U_{\max} = \max\{U(\omega)\}$; $Y_n(\omega_1,\cdots,\omega_n)$ is the output from *n*-th order kernel which form a Hilbert space. Each of the output terms Y_1, Y_2, … and Y_n take on different dimensions, which consist of different number of dependent variables. The higher order terms can be converted into their one-dimensional equivalents using dimensional contraction technique by taking a multi-dimensional convolution across all variables. Thus (13) can be rewritten as

$$Y(\omega) = Y_1(\omega) + \int_{-\infty}^{\infty} Y_2(\omega_1, \omega-\omega_1) d\omega_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} Y_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2) d\omega_1 d\omega_2 + \cdots$$

$$= H_1(\omega)U(\omega) + \int_{-\infty}^{\infty} H_2(\omega_1,\omega-\omega_1) U(\omega_1) U(\omega-\omega_1) d\omega_1 + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2)$$

$$\times U(\omega_1) U(\omega_2) U(\omega-\omega_1-\omega_2) d\omega_1 d\omega_2 + \ldots \tag{14}$$

Utilising the triangular inequality property of the Hilbert space, the upper bound for the output can be obtained as

$$|Y(\omega)| \le |H_1(\omega, z)| \times |U_{\max}| + \left|\int_{-\infty}^{\infty} H_2(\omega_1,\omega-\omega_1) d\omega_1\right| \times |U_{\max}|^2$$

$$+ \left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2) d\omega_1 d\omega_2\right| \times |U_{\max}|^3 + \ldots \tag{15}$$

The convergence for the Volterra series can be guaranteed if and only if the infinite power-series on the right hand side of the inequality is convergent. Accordingly (15) leads to

$$\frac{\left|\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{n+1} H_{n+1}(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_n)d\omega_1\ldots d\omega_n\right|}{\left|\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{n} H_n(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n-1})d\omega_1\ldots d\omega_{n-1}\right|}\times\left|U_{\max}\right|<1 \tag{16}$$

Since the even order kernels are null in a single-mode fibre, the convergence criteria (16) can be written as

$$\frac{\left|\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{n+2} H_{n+2}(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n+1})d\omega_1\ldots d\omega_{n+1}\right|}{\left|\underbrace{\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty}}_{n} H_n(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n+1})d\omega_1\ldots d\omega_{n-1}\right|}\times\left|U_{\max}\right|^2<1 \tag{17}$$

where n is an odd positive integer. Hence the upper-bounds for the inputs to each kernel of different orders can be expressed in terms of the integration of lower-order kernels. This expression can be simplified to, under the case of $n = 1$ or the convergence of the third order kernel with respect to the linear kernel as

$$\frac{\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2)d\omega_1 d\omega_2\right|}{\left|H_1(\omega)\right|}\times\left|U_{\max}\right|^2<1 \tag{18}$$

that is $$\left|H_1(\omega)\right|>\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2)d\omega_1 d\omega_2\right|\times\left|U_{\max}\right|^2 \tag{19}$$

Indeed (19) indicates the relationship between the linear transfer function or the effective bandwidth of an optical fibre operating in the linear dispersion region and the dispersion effect due to the self phase modulation due to the intense optical pulse power contributing via the third order kernel.

Therefore for the case of $n = 3$, the expression for upper-bounds can now be obtained as a relationship between the 5th order nonlinear coefficients of the fibre such as the four wave mixing effects and that of the self-phase modulation, the 3rd order kernel function of the VSTF. The upper-bounds for inputs to higher order kernels can be derived in the same way.

$$\frac{\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega-\omega_1-\omega_2-\omega_3-\omega_4)d\omega_1 d\omega_2 d\omega_3 d\omega_4\right|}{\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_3(\omega_1,\omega_2,\omega-\omega_1-\omega_2)d\omega_1 d\omega_2\right|}\times\left|U_{\max}\right|^2 < 1 \qquad (20)$$

can be replaced by

$$\frac{\left|\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_5(\omega_1,\omega_2,\omega_3,\omega_4,\omega-\omega_1-\omega_2-\omega_3-\omega_4)d\omega_1 d\omega_2 d\omega_3 d\omega_4\right|\times\left|U_{\max}\right|^4}{\left|H_1(\omega)\right|} < 1 \qquad (21)$$

Here we denote the norm of the *n*th order kernel by

$$\delta_n = \sup\left\{\frac{\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} H_n(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n-1})d\omega_1\ldots d\omega_{n-1}\right|}{\left|H_1(\omega)\right|}\right\} \qquad (22)$$

In general, the radius of convergence as a function of the total input optical field amplitude can therefore be expressed as

$$U_{\max} = \left(\frac{1}{\delta_n}\right)^{\frac{1}{n-1}} = \inf\left\{\left(\frac{\left|H_1(\omega)\right|}{\left|\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} H_n(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n-1})d\omega_1\ldots d\omega_{n-1}\right|}\right)^{\frac{1}{n-1}}\right\} \qquad (23)$$

Accordingly, the *peak power* of the input pulse so that the VSTF is convergent and hence computable is given by

$$P_{peak} = U_{\max}{}^2 = \left(\frac{1}{\delta_n}\right)^{\frac{2}{n-1}} = \inf\left\{\left(\frac{|H_1(\omega_1)|}{\left|\int_{-\infty}^{\infty}\ldots\int_{-\infty}^{\infty} H_n(\omega_1,\ldots,\omega-\omega_1-\ldots-\omega_{n-1})d\omega_1\ldots d\omega_{n-1}\right|}\right)^{\frac{2}{n-1}}\right\} \quad (24)$$

This is the most important result and can be used in the determination of the linear and nonlinear operation of an optical fibre communications under high and intense optical input pulses, especially when there exists numerous optical channels in a single fibre. Furthermore the radius of convergence of the higher order of the VSTF indicate the level of manageable of the input optical pulse power so that the linear dispersion effect can be compensated by the nonlinear effects in the fibre. Otherwise the series would be divergent and hence the radiation or complete depletion of the optical pulses due to dispersion.

V. ESTIMATION OF OPTICAL INPUT POWER FOR GUIDING OVER A FIBRE LENGTH

In this section, numerical results of the kernels of the transfer functions expressed via the Volterra series are presented for a fibre with a linear attenuation coefficient of $\alpha_0 = 0.2$ dB/km at the operating wavelength of 1.55 µm, $\beta_2 = -1.2746 \times 10^{-27}$ corresponding to a group velocity dispersion = 2.0 ps/km-nm, and $\beta_3 = 0$ ie. zero dispersion slope; the nonlinear refractive index assumes the normal value for silica fibre $n_2 = 2.31 \times 10^{-20}$ m^2/W with an effective cross-sectional core area of $A_{eff} = 80$ µm^2.

As described in previous section, the radius of convergence (ROC) of the nth order kernels of the VSTF governs the upper bounds for the maximum input power, and is correspondent to their norm δ. The norm δ_n for 3rd, 5th, and 7th order kernels are thus calculated with the fibre length as a parameter. The radius of convergence could then be computed under these conditions leading to a relationship between the optical peak power and the fibre length. The limits of integration of the kernels in computation of respective norms are restricted to the frequency range of the spectrum that can be severely affected by the two adjacent optical carrier channels in standard ITU grid of 100GHz spacing for 10 Gbps channels.

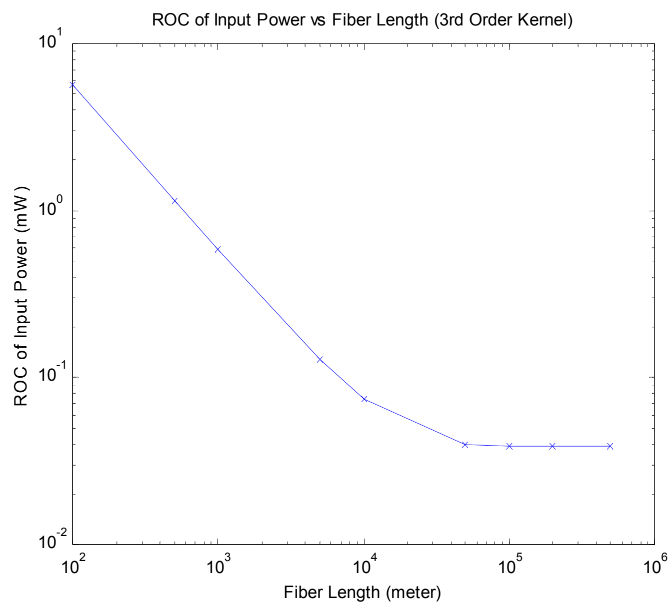

Figure 1. Input pulse peak power as a function of fibre length of the 3rd order transfer function.

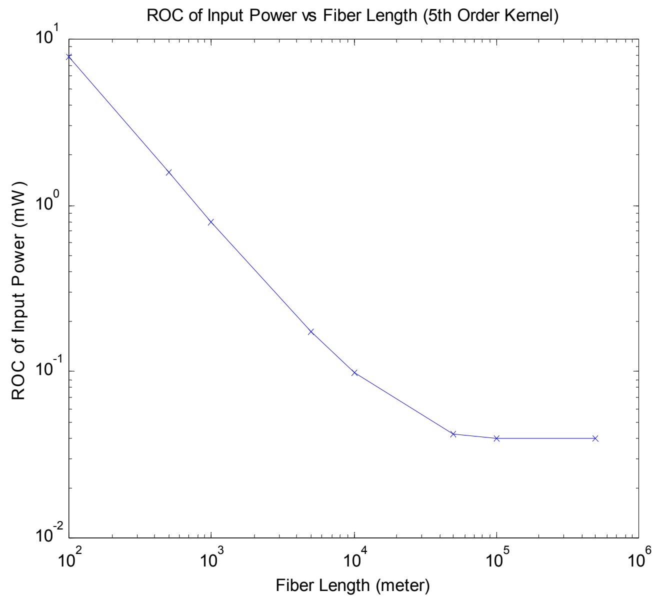

Figure 2: Input pulse peak power as a function of the fibre length of the 5th order transfer function.

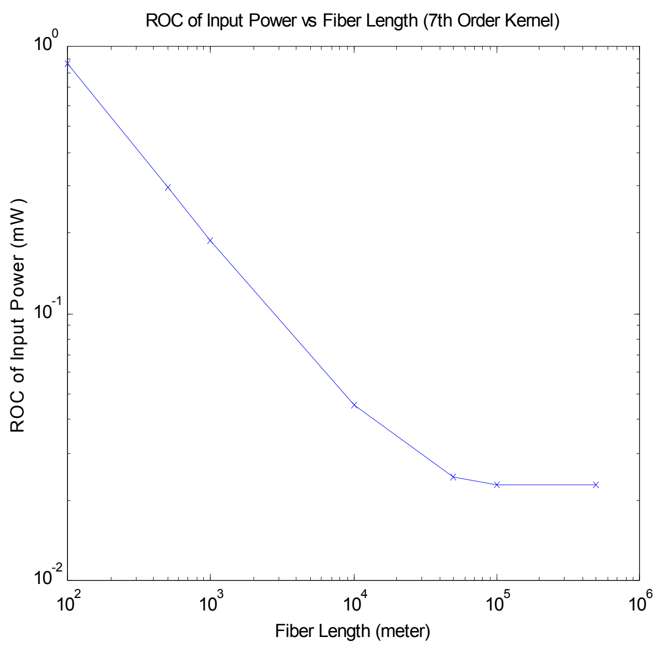

Figure 3: Input pulse peak power as a function of fiber length of the 7th order transfer function.

Figures 1, 2 and 3 show the dependence of the input pulse optical peak power evaluated using the radii of convergence of the 3^{rd}, 5^{th} and 7^{th} order kernels respectively of the Volterra series as a function of the total transmission length of the fibre. A similar pattern for relationship between the fibre length and the radius of convergence for the input power has been observed. The ROC varies inverse-proportionally with the fibre length over a significant distance as displayed in the log-log scale. An interesting phenomenon is that the decrease of ROC begins to slow down when the fibre length approaches 50 km, and remains roughly constant after the fiber length is beyond 100 km. This result agrees with [1] in that with input power of 10 nW and 1 mW, the errors are small; while with input power of 30 mW, which is close to the upper bound, the error is of several magnitude larger.

A. Comparison between VSTF and SSF methods

The proposal of the computation of radius of convergence can be further supported by the comparison between results obtained with VSTF and SSF method. In this simulation, input pulses of Gaussian shape with different peak power are propagated through a fiber length of 100 km. The difference is calculated by setting a criterion

$$Deviation \quad in \ \% = \frac{\int_{-\infty}^{\infty} \left| U_2(\omega) - U_1(\omega) \right|^2 d\omega}{\int_{-\infty}^{\infty} \left| U_2(\omega) \right|^2 d\omega} \tag{25}$$

where $U_1(\omega)$ is the output spectrum from the VSTF method while $U_2(\omega)$ is the output spectrum obtained from the SSF method.

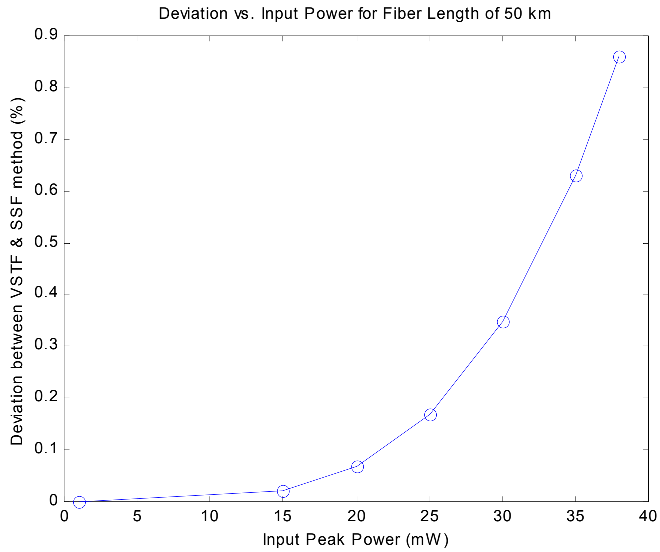

Figure 4. Deviation factor as a function of input optical power for a fibre length of 50 km

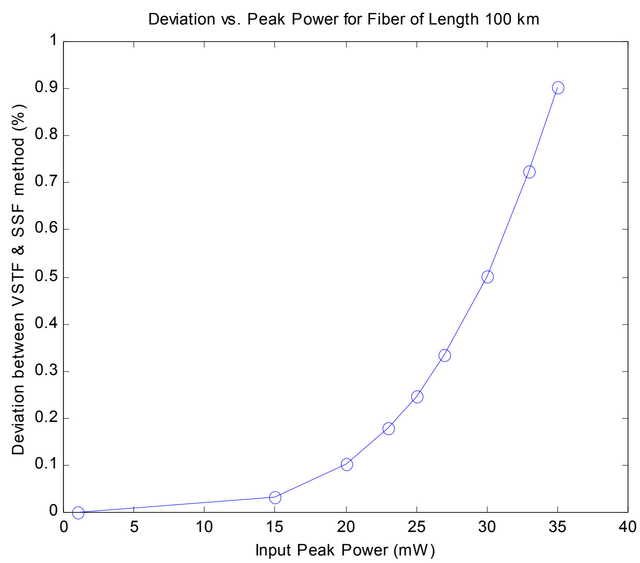

Figure 5. Deviation factor as a function of input power for a fibre length of 100 km

Figures 4 and 5 show the relationship between the deviation and the input peak optical power of pules after transmission over a fibre length of 50 km and 100 km respectively. As the peak power increases and approaches the upper-bound (ROC), the deviation between results obtained VSTF and SSF increases quickly.

B. Application of the VSTF to solitonic optical transmission

Historically, optical communication development can be classified into different generations[7] according to the operating regions of the transmission channel in the linear or nonlinear regime with the total power is the effective power as seen by the fibre with the superposition of all individual optical channels. The current state-of-art systems operate at over a very wide wavelength regions from 1480 nm to 1620 nm, utilizing wavelength-division-multiplexing (WDM), with dispersion compensated or managed composite transmission fibres. Optical fibre communication systems concern with the balancing of the fibre linear dispersion (GVD) and the self phase variation due to nonlinear effect. Indeed optical solitons preserve their shape during propagation in a lossless fiber by counteracting the effect of dispersion via the shaping of the fibre nonlinearity. We demonstrate the effectiveness of the Volterra series approach to the transmission of a soliton of fundamental order through a 50 km long fiber. Further both VSTF and SSF methods are compared for the evolution of the propagated pulses.

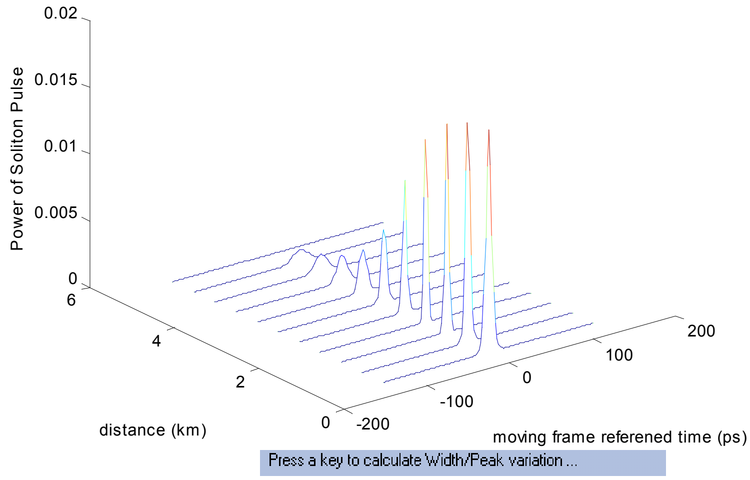

Figure 6 Solitonic pulses propagation via the simulation by SSF method

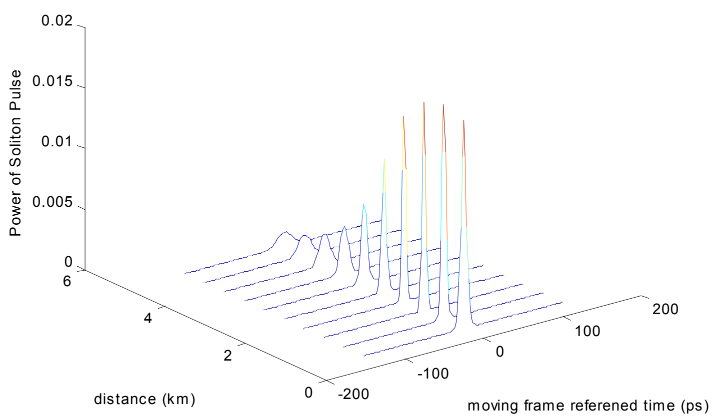

Figure 7 Solitonic pulses propagation via the simulation by VSTF method

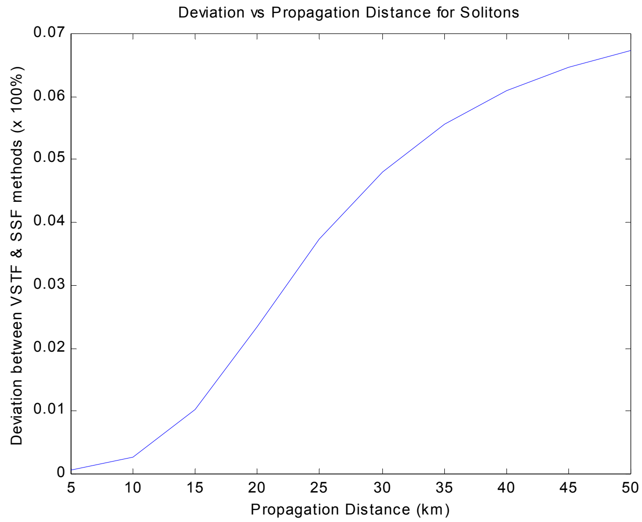

Figure 8. Deviation factor between VSTF and SSF methods

It is well known that the power requirement for nonlinear optical pulses under complete compensation between the linear dispersion and the self phase modualtion, usually known as solitons, is governed by the soliton order, its pulse width, the pulse peak power and the dispersion factor of the transmission fibre. In this simulation, the following parameters are used: soliton order = 1, pulse width = 10 ps (T_{fwhm}), GVD = 1 ps/(km-nm). Accordingly, a power P_0 of 16.923 mW is required for soliton formation. Generally, the power required by soliton systems are much higher than conventional ones since the power required to generate nonlinearity to counter the dispersion effect is substantial. Figures 6 and 7 show the transmission of such first order solitonic pulses of fundamental order by using the SSF method and the VSTF method under the convergence condition for the 3rd kernel transfer function as described in previous section, equation (24).

VI. CONCLUDING REMARKS

The Volterra series transfer function has been effectively employed to model the wave propagation inside single-mode fibers. Both linear and nonlinear effects can practically be approximated up to 7th order depending on the accuracy required. This paper provides a conservative criterion for calculating the upper bounds of the input power under which the VSTF model remains convergent. The relationship between radius of convergence for input power and the fiber length is studied. It is shown that the ROC varies inverse-proportionally with the fiber length logarithmically. However, as the fiber length increases beyond 100 km, the ROC for input power approximately remains roughly unchanged. Therefore the VSTF model remains valid even for fiber of very long length, provided the input power is below the upper bound of the ROC. Since the launched power in optical communication systems is typically below 10 mW, and the distances between amplifiers are less than 100 km, the VSTF model is quite adequate under normal operating condition.

Practically, the VSTF method is more suitable for investigating the frequency response of an optical communication system, and hence appropriate for system design. However, it is not as efficient as the SSF method in simulating pulse evolution inside fibers unless the propagation distance is relatively long.

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