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Abstract

We derive divergence-free vector wavelets from any scalar wavelet. The derived vector wavelet transform is equal to the restriction, to the class of divergence-free wavelets, of the scalar wavelet transform applied to each component of a vector function. This means that the new wavelet inherits all of the nice properties of the scalar wavelet from which it is derived. The approach is thus not only far simpler but it is far more powerful than any other approach yet devised.

1 Introduction

The approximation or representation of divergence-free vector functions has many applications such as the study of certain types of fluids or other incompressible materials, and the study of certain vector fields such as sourceless electric fields. Battle [BF93] and Lemarie-Rieusset [LR91] have both derived divergence-free vector wavelets. However, in contrast to their approaches, ours is a rather simple translation of a scalar wavelet into a vector wavelet.

We consider, for simplicity, vector functions whose components belong to $L^2(R^n)$. To make the arguments as transparent as possible, we will rely on the properties of the usual Fourier transform, in particular that of the isometry $< f, g > = < \hat{f}, \hat{g} >$ where $< \cdot, \cdot >$ is the usual inner product $\int f\overline{g}\, dx$ on $L^2(R^n)$. However, our arguments should generalise to other spaces and little changes should need to be made other than the replacement with the relevant Fourier transform type isometry.

2 Vector Wavelets

Firstly, we recall the wavelet decomposition of $L^2(R^n)$ (see for example [Dau92], section 2.6). We can choose a spherically symmetric wavelet $\psi \in L^2(R^n)$, $\psi(x) = \eta(|x|)$, which satisfies the admissability condition:

$$C_\psi = (2\pi)^n \int_0^\infty \frac{dt}{t} |\eta(t)|^2 < \infty. \tag{1}$$

It is then possible to prove that the wavelet transform $W_\psi(f)(a,b) = < f, \psi^{a,b} >$, formed by dilates (parameterised by a) and translates (parameterised by b) of ψ, $\psi^{a,b}(x) = a^{-\frac{n}{2}} \psi(\frac{x-b}{a})$, has the following property:

$$f = C_\psi^{-1} \int_0^\infty \frac{da}{a^{n+1}} \int_{R^n} db W_\psi(f)(a,b) \psi^{a,b}. \tag{2}$$

This property being the ability to recover the original signal from the wavelet decomposition. We note, but for simplicity do not address further, that if the wavelet is not rotationally symmetric, then one may parameterise the transform with rotations of ψ.

In our vector spaces $(L^2(R^n))^n$, we define a scalar product $< u, v >_{(L^2(R^n))^n} = < u_1, v_1 > + \ldots + < u_n, v_n >$. However, we will drop the subscript on this inner product since we can distinguish between the inner products on the basis of the the arguments to the inner products. We also replace the scalar function ψ with an $n \times n$ matrix function Ψ whose row vectors $\{\Psi^i : i = 1 \ldots n\}$, can be thought of as (the transpose of) vector wavelets. We also define a bilinear form:

$$< \Psi, v > = \begin{pmatrix} < \Psi^1, v > \\ < \Psi^2, v > \\ \vdots \\ < \Psi^n, v > \end{pmatrix} \tag{3}$$

By an abuse of notation, we use the same symbol for the $L^2(R^n)$ inner product and for the inner products we have so far defined. The vector wavelet transform $\Psi^{a,b}(v)$ is, itself, an n vector.

Now if Ψ is a diagonal matrix, $\mathrm{diag}(\psi_1, \ldots, \psi_n)$, then the vector wavelet transform becomes a set of n separate scalar transforms:

$$< \Psi, v > = \begin{pmatrix} < \psi_1, v_1 > \\ < \psi_2, v_2 > \\ \vdots \\ < \psi_n, v_n > \end{pmatrix} \tag{4}$$

applied independently to each scalar component of the vector v. In particular, we could employ the same scalar wavelet $\psi_i = \psi$. This is the most obvious approach to the wavelet decomposition of a vector function but it is not particularly suited to vector functions that are divergence-free. The reason for this will now be sketched.

3 Divergence-Free Wavelets

The subspace of divergence-free vector fields, contained in $(L^2(R^n))^n$, is a convex subspace with no interior (see for example [You87]). The importance of this is that there are vector fields in $(L^2(R^n))^n$ that are arbitrarily close to the set but which have very large divergences (consider adding the vector field $\epsilon(\sin wx, \sin wy)$ to any member of the divergence-free 2-D vector fields, by taking the constant ϵ small we can make the new vector field arbitrarily close to the divergence-free vector field but by making the constant w arbitrarily large we can make this vector field have arbitrarily large divergence). This fact has significance when one performs numerical work. Small numerical errors in computation or measurement can produce a vector function with large divergence when the "true" function should have divergence zero. The solution is to develop a method of approximation/representation that projects onto the divergence-free functions.

Now the process of taking a wavelet transform is a projection: indeed, it is well known that the projection, using the affine group, as is usual, and taking functions from $L^2(R)$, projects not onto the whole of $L^2(R)$ but onto a dense subset (i.e., the range of the wavelet transform). Similar statements hold for other spaces and other groups defining the wavelets (see, for example, [Hol93]). However, if individual wavelet transforms are applied to each component of a vector field separately, we will not be able to guarantee that we will project onto a set of divergence-free vector fields within the range of the wavelet transform. What we want to achieve, is to derive a set of wavelet transforms, related to a given scalar wavelet transform, that projects onto the the divergence-free subset. If we do so in a manner that this new transform agrees, on this subset, with the component-wise scalar transforms when restricted to this subset, then we will be able to inherit from the scalar transforms all of the nice properties (such as stable reconstruction). It is this idea that makes out approach attractive and simple compared to the developments published so far.

Our approach will produce a wavelet decomposition for vector fields of any dimension but for simplicity we restrict ourselves to 2 and 3 dimensions (these cover the majority of engineering calculations). Given a scalar wavelet ψ for $L^2(R^2)$ or $L^2(R^3)$, then the following matrices define divergence-free wavelet decompositions:

$$\hat{\Psi} = \frac{\hat{\psi}}{\xi_1^2+\xi_2^2+\xi_3^2}\begin{pmatrix} \xi_2^2+\xi_3^2 & -\xi_1\xi_2 & -\xi_1\xi_3 \\ -\xi_2\xi_1 & \xi_1^2+\xi_3^2 & -\xi_2\xi_3 \\ -\xi_1\xi_3 & -\xi_2\xi_3 & \xi_1^2+\xi_2^2 \end{pmatrix} \tag{5}$$

for $(L^2(R^3))^3$

$$\hat{\Psi} = \frac{\hat{\psi}}{\xi_1^2+\xi_2^2}\begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_2\xi_1 & \xi_1^2 \end{pmatrix} \tag{6}$$

for $(L^2(R^2))^2$. We use the symbols ξ_i for the Fourier frequency variables. In either of the two cases, we need to clarify the definition at the origin in Fourier space (denominator vanishes). Clearly the origin represents the "dc" signal which is divergence free and hence the projection becomes ψI, where I is the $n \times n$ identity.

Using the Fourier pairs $\frac{\partial}{\partial x}$ and $-i\xi_1$, it is easy to interpret the matrices in the "real domain". Namely, we consider any function κ such that $\Delta\kappa = \psi$ and then the divergence-free wavelets are defined by the matrices:

$$\Psi = \begin{pmatrix} \frac{\partial^2\kappa}{\partial y^2}+\frac{\partial^2\kappa}{\partial z^2} & -\frac{\partial^2\kappa}{\partial x\partial y} & -\frac{\partial^2\kappa}{\partial x\partial z} \\ -\frac{\partial^2\kappa}{\partial x\partial y} & \frac{\partial^2\kappa}{\partial x^2}+\frac{\partial^2\kappa}{\partial z^2} & -\frac{\partial^2\kappa}{\partial y\partial z} \\ -\frac{\partial^2\kappa}{\partial x\partial z} & -\frac{\partial^2\kappa}{\partial y\partial z} & \frac{\partial^2\kappa}{\partial x^2}+\frac{\partial^2\kappa}{\partial y^2} \end{pmatrix} \tag{7}$$

for $(L^2(R^3))^3$

$$\Psi = \begin{pmatrix} \frac{\partial^2\kappa}{\partial y^2} & -\frac{\partial^2\kappa}{\partial x\partial y} \\ -\frac{\partial^2\kappa}{\partial x\partial y} & \frac{\partial^2\kappa}{\partial x^2} \end{pmatrix} \tag{8}$$

for $(L^2(R^2))^2$. The method of deriving the matrices for any dimension is sketched in the appendix.

Of course, it may be difficult to find an explicit expression for κ, but we can always perform calculations in the Fourier domain. Indeed, for the rest of the paper we prefer to base our developments on the Fourier domain representations.

A simple calculation shows that the rows of our matrices are, indeed, divergence-free vector fields. To show that the wavelet transform, defined by these matrices, agrees with the component-wise scalar transforms when restricted to the set of divergence-free vector fields, we use the fact that the wavelet transform is a series of inner products and that $< u, v > = < \hat{u}, \hat{v} >$. For brevity we consider only the 2-D case, the 3-D case follows in a similar manner.

Thus, consider the calculation of the wavelet coefficient (vector) corresponding to $\psi^{a,b}$. We have:

$$< \hat{\Psi^{a,b}}, (\hat{u}, \hat{v}) >= \frac{\hat{\psi^{a,b}}}{\xi_1^2+\xi_2^2} \begin{pmatrix} \xi_2^2\overline{\hat{u}} - \xi_1\xi_2\overline{\hat{v}} \\ -\xi_1\xi_2\overline{\hat{u}} + \xi_1^2\overline{\hat{v}} \end{pmatrix} \tag{9}$$

But if the divergence of (u, v) is zero then we have $\xi_1 \hat{u} = -\xi_2 \hat{v}$ which we can use to eliminate $\hat{v}$ from the first row of the right hand side and $\hat{u}$ from the second row. That is:

$$< \hat{\Psi^{a,b}}, (\hat{u}, \hat{v}) >= \begin{pmatrix} < \hat{\psi^{a,b}}, \hat{u} > \\ < \hat{\psi^{a,b}}, \hat{v} > \end{pmatrix} \tag{10}$$

so the new vector wavelet transform agrees with the restriction of the component-wise scalar transforms.

It is instructive to relate our approach to the various versions of the Helmholtz decomposition theorem which states that, under certain conditions, a vector field μ can be decomposed into two parts: $\mu = \mu_d + \mu_c$ where μ_d is divergence-free and μ_c is curl-free. Although the decomposition is not unique (there are vectors fields, constant, that are divergence-free and curl-free) we can assume w.l.o.g, that the decomposition is performed so that all of the divergence-free parts are contained in μ_d. In essence, we are projecting to zero the second part of the vector field. To see this, it is again a simple matter of applying the matrices we have developed, in the Fourier domain, and using the constraint that vector field is curl-free (e.g., in 2-D, $\mu_c = (\mu_1, \mu_2)$, we have $\xi_1\mu_2 = \xi_2\mu_2$). It is then easily seen that this component is projected to the zero vector. Moreover, our sketch of the derivation of the matrices (appendix), relating as it does, the divergence to radial components in Fourier space and curl to tangential components provides some insight into why, up to a constant (i.e., the origin in Fourier space, where the polar coordinate system breaks down) the divergence and curl characterises the vector field.

4 Conclusion

The fact that our divergence-free transform projects onto divergence-free vector fields but agrees with the component-wise scalar transforms restricted to that set, brings a great many properties. Firstly, if the scalar transform is a Riesz basis or a frame for $L^2(R^n)$, then our vector transform must also be a Riesz basis or a frame for $(L^2(R^n))^n_{div-free}$. This means we can develop stable discrete transforms when the scalar wavelet provides a stable discrete transform. Moreover, if our scalar transform has nice properties such as compactness, or vanishing moments, since our vector transform agrees on $(L^2(R^n))^n_{div-free}$ we will also benefit from these properties. Indeed, we need not even apply a special inverse transform, since the wavelet coefficients we obtain are precisely those we obtain by taking the scalar-component-wise transform on the divergence-free part of the vector field. However, it is possible that numerical errors will again produce some degree of divergence in the inversion process and so it is probably advisable to use our vector divergence-free wavelets in the inversion process as well. Though we haven't specifically addressed the problem, the extension of these ideas to curl-free approximations by wavelets is straightforward.

We conclude by noting that the same methods can be used to derive divergence-free approximations of other kinds, if the process essentially relies on a type of inner product. In particular, we note that our methods can be used to derive spline kernels such as [AB91] or [Han93]. More importantly, the method generalises to other splines not based upon the thin-plate of polyharmonic kernels. These matters, and, indeed, the study of other approximation methods such as finite elements, in the light of our method, is the subject of a paper in preparation.

A Derivation

We sketch the motivation/derivation behind our vector wavelets. For simplicity we consider only the 2-D case, the 3-D case being a straightforward generalization. We realise the the essence of the wavelet transform is to project by scalar products with the wavelet. Now consider an arbitrary vector field (u, v) in $L^2(R^n)$ with Fourier transform $(\hat{u}, \hat{v})$. The divergence of this vector field has Fourier transform $(i\xi_1\hat{u}, i\xi_2\hat{v})$ which is just the radial component (up to the factor i) of the Fourier transformed vector fields - see figure 1. Since we want to project onto divergence-free vectors, we want to remove this radial component and preserve the other (tangential) component. This can be done by rotating the vectors (locally) to align the radial component with the ξ_1 axis and then projecting using the matrix:

$$\begin{pmatrix} 0 & 0 \\ 0 & \hat{\psi} \end{pmatrix} \tag{11}$$

and then rotating back. In other words we perform a local change of basis, project, and then revert back to the old basis. It is clear that the rotation matrix is:

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} = \begin{pmatrix} \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} \\ -\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} \end{pmatrix} \tag{12}$$

Applying the rotations we derive:

$$\begin{pmatrix} \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & -\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} \\ \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & \hat{\psi} \end{pmatrix} \begin{pmatrix} \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} \\ -\frac{\xi_2}{\sqrt{\xi_1^2+\xi_2^2}} & \frac{\xi_1}{\sqrt{\xi_1^2+\xi_2^2}} \end{pmatrix} = \frac{\hat{\psi}}{\xi_1^2+\xi_2^2} \begin{pmatrix} \xi_2^2 & -\xi_1\xi_2 \\ -\xi_2\xi_1 & \xi_1^2 \end{pmatrix} \tag{13}$$

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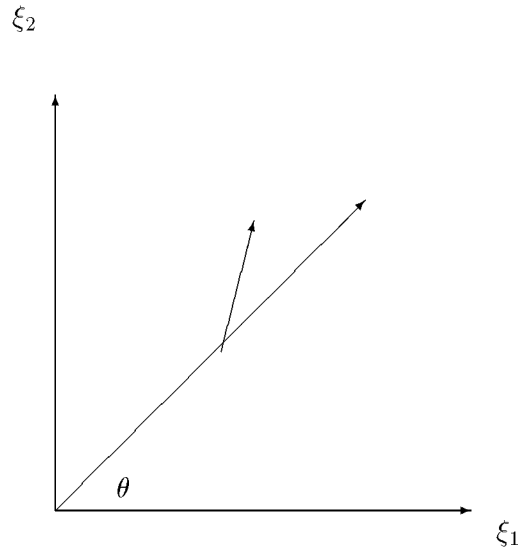

Figure 1: Fourier Plane Representation

We consider the value of the vector field at an arbitrary point (ξ_1, ξ_2) - small vector. The component of this field with non-zero divergence is the component that is oriented in the radial direction (i.e., the projection of the small vector onto the large vector).